Functional renormalization group and its applications to gauge theories

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Outline

Motivation

- Quantum field theories are extensively used in particle physics, condensed matter physics, statistical mechanics...
  $\longrightarrow$ $3+1$ dim. continuum theories
  $\longrightarrow$ thermal field theories
  $\longrightarrow$ dimensionally reduced effective models
- Central objects that one is looking for:
  $\langle \phi_1 \phi_2 ... \phi_n \rangle$ correlation functions

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- Quantum field theories are extensively used in particle physics, condensed matter physics, statistical mechanics...
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  $\longrightarrow$ thermal field theories
  $\longrightarrow$ dimensionally reduced effective models
- Central objects that one is looking for:
  $\langle \phi_1 \phi_2 ... \phi_n \rangle$ correlation functions
- Most successful method: perturbation theory
  $\longrightarrow$ Taylor-expansion based on some small parameters
  $\longrightarrow$ asymptotic series, does not necessarily show any convergence
- Non-perturbative methods are necessary!
  $\longrightarrow$ Functional Renormalization Group (FRG)

Wilsonian and Functional Renormalization

- At the critical point in 2nd order phase transitions: massless modes invalidate PT (IR singularities)
- Idea of the Wilsonian Renormalization Group:
  $\longrightarrow$ momentum-shell integration: introduce an intermediate scale k and define
  $\phi(\vec{p}) = \phi^{<}(\vec{p})\Theta(k-|\vec{p}|) + \phi^{>}(\vec{p})\Theta(|\vec{p}|-k)$

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi^{<} \int \mathcal{D}\phi^{>} e^{-S_{<}-S_{>}-S_{mix}[\phi^{<},\phi^{>}]}$$

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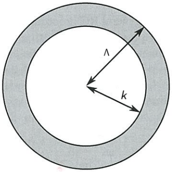
$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi^< \int \mathcal{D}\phi^> e^{-S_< - S_> - S_{mix}[\phi^<,\phi^>]}$$

- $\phi^>$ integral is performed using pert. theory

$$Z = \int \mathcal{D}\phi^< e^{-S[\phi^<]}$$

$$S \to S - \log <e^{-[S_>] - S_{mix}[\phi^<,\phi^>]}>$$



- Result: $\Lambda \to k$, $m_\Lambda^2 \to m_k^2$, $g_\Lambda \to g_k$ + new vertices

Wilsonian and Functional Renormalization

- Rescaling dimensionful quantities with k
  $\longrightarrow$ flow equations for individual coupling constants:

$$k\partial_k \bar{m}_k^2 = \beta_m(\bar{m}_k^2, \bar{g}_k^{(i)}, \ldots) \qquad k\partial_k \bar{g}_k^{(i)} = \beta_{g^{(i)}}(\bar{m}_k^2, \bar{g}_k^{(i)}, \ldots)$$

- Fixed points: statistically self-similar behavior
  $\rightarrow$ no relevant length scale! (2nd order phase transitions)
- Identification of relevant and irrelevant directions
  $\rightarrow$ close to a fixed point only the relevant ones matter
  $\rightarrow$ finite T transition: one relevant direction

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- The WRG has had great success in
  $\longrightarrow$ describing universality in 2nd order transitions
  $\longrightarrow$ predicting critical exponents
  $\longrightarrow$ understanding the concept of effective theories

Wilsonian and Functional Renormalization

- FRG generalizes the idea of the WRG: fluctuations are taken into account at the level of the quantum effective action

$$Z[J] = \int \mathcal{D}\phi e^{-(\mathcal{S}[\phi]+\int J\phi)} \quad \Rightarrow \quad \Gamma[\bar{\phi}] = -\log Z[J] - \int J\bar{\phi}$$

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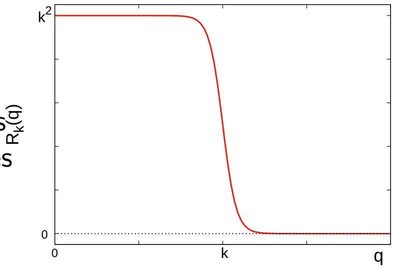
- Introduction of a flow parameter k and inclusion of fluctuations for which $q \gtrsim k$

$$Z_k[J] = \int \mathcal{D}\phi e^{-(\mathcal{S}[\phi]+\int J\phi)} \times e^{-\frac{1}{2}\int \phi R_k \phi}$$

$\longrightarrow$ regulator: mom. dep. mass term suppressing low modes

$\longrightarrow$ not only sharp cutoff!



Wilsonian and Functional Renormalization

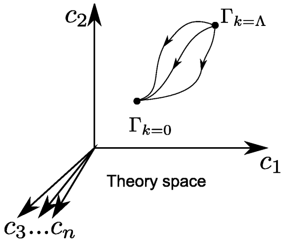
- Scale-dependent effective action:

$$\Gamma_k[\bar\phi] \quad = \quad -\log Z_k[J] - \int J\bar\phi - \frac{1}{2}\int \bar\phi R_k \bar\phi$$

$\longrightarrow k \approx \Lambda$: no fluctuations $\Rightarrow \Gamma_{k=\Lambda}[\bar\phi] = \mathcal{S}[\bar\phi]$
$\longrightarrow k = 0$: all fluctuations $\Rightarrow \Gamma_{k=0}[\bar\phi] = \Gamma[\bar\phi]$

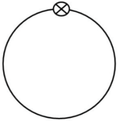
- The scale-dependent effective action interpolates between classical- and quantum effective actions

- The trajectory depends on R_k but the endpoint does not



- Flow of the effective action is described by the Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2} \int_{qp} \mathrm{Tr} \left[(\Gamma_k^{(2)} + R_k)^{-1}(p,q) \partial_k R_k(-q,-p) \right] = \frac{1}{2}$$



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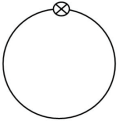
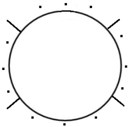


- Slightly different form: [$\tilde{\partial}_k$ acts only on R_k]

$$\partial_k \Gamma_k = \frac{1}{2}\int \tilde{\partial}_k\, \mathrm{Tr}\, \log[\Gamma_k^{(2)} + R_k] = \frac{1}{2}\tilde{\partial}_k \sum$$



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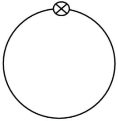

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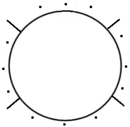

- Scale dependence of the proper vertices are described by one-loop diagrams [propagators are dressed!]

  $\longrightarrow$ exact equation!

- Advantage: flows are directly accessible in any dimension
  but approximation is needed

Field theoretical RG vs. the FRG

- How is the Wilsonian (and Functional) RG related to the field theoretical RG?
- Considering the Euclidean ϕ^4 theory in 4d:

$$\Gamma^{(2)}(0) = m^2 + \delta m^2 - $$  $$ + \ldots = m^2 + \frac{\lambda}{2}\int_q \frac{1}{\vec{q}^{\,2}+m^2} + \ldots$$

$$\Gamma^{(4)}(0) = \lambda + \delta\lambda - $$  $$ + \ldots = \lambda - \frac{3\lambda^2}{2}\int_q \frac{1}{(\vec{q}^{\,2}+m^2)^2} + \ldots$$

- Minimal substracion:

$$\delta m^2(\mu) = -\frac{\lambda}{2}\int_q \left(\frac{1}{\vec{q}^{\,2}+\mu^2} + \frac{\mu^2-m^2}{(\vec{q}^{\,2}+\mu^2)^2}\right)$$

$$\delta\lambda(\mu) = \frac{3\lambda^2}{2}\int_q \frac{1}{(\vec{q}^{\,2}+\mu^2)^2}$$

Field theoretical RG vs. the FRG

- Finite 2 and 4-point functions:

$$\Gamma^{(2)}(0) = m_\mu^2 + \frac{\lambda}{2}\int_q \left(\frac{1}{\vec{q}^{\,2}+m_\mu^2} - \frac{1}{\vec{q}^{\,2}+\mu^2} - \frac{\mu^2-m_\mu^2}{(\vec{q}^{\,2}+\mu^2)^2}\right) + \dots$$

$$\Gamma^{(4)}(0) = \lambda_\mu - \frac{3\lambda^2}{2}\int_q \left(\frac{1}{(\vec{q}^{\,2}+m_\mu^2)^2} - \frac{1}{(\vec{q}^{\,2}+\mu^2)^2}\right) + \dots$$

- Integrand of the fluctuation contributions cuts off at $q \sim \mu$
  $\longrightarrow$ m_μ^2 and λ_μ plays the role of m_k^2 and λ_k!
  $\longrightarrow$ Why? $\Rightarrow$ m_k^2 and λ_k refer to an effective action where fluctuations are included where $q \gtrsim k$

- Renormalization scale μ in the field theoretial RG

  $\equiv$

  Separation scale k in the Wilsonian (Functional) RG

Field theoretical RG vs. the FRG

- Two approaches to obtain the β functions:

  $\longrightarrow$ β_{m^2}: $k\partial_k m_k^2$ or $\mu\partial_\mu m_\mu^2$

  $\longrightarrow$ β_λ: $k\partial_k \lambda_k$ or $\mu\partial_\mu \lambda_\mu$

- Field theoretical RG:

$$\mu\partial_\mu m_\mu^2 = -\frac{\lambda_\mu}{2}(\mu^2 - m_\mu^2), \quad \mu\partial_\mu \lambda_\mu = \frac{3\lambda_\mu^2}{2}$$

- Wilsonian (Functional) RG:

$$k\partial_k m_k^2 = -\frac{\lambda_k}{2}\frac{k^4}{k^2+m_k^2}, \quad k\partial_k \lambda_k = \frac{3\lambda_k^2}{2}\frac{k^4}{(k^2+m_k^2)^2}$$

- Two results agree but only close to the Gaussian fixed point!

  $\longrightarrow$ Wilsonian (Functional) RG is more general!

Field theoretical RG vs. the FRG

- Why do we use the field theoretical RG after all?
  $\longrightarrow$ it does not necessarily introduce a momentum cutoff

  (dim. reg., Pauli-Villars, etc.)

  $\Longrightarrow$ gauge symmetry survives

  $\longrightarrow$ QCD in the UV and QED in the IR works fine
  (small couplings)

Field theoretical RG vs. the FRG

- Why do we use the field theoretical RG after all?
  $\longrightarrow$ it does not necessarily introduce a momentum cutoff
  (dim. reg., Pauli-Villars, etc.)
  $\Longrightarrow$ gauge symmetry survives
  $\longrightarrow$ QCD in the UV and QED in the IR works fine
  (small couplings)
- Wilsonian (Functional) RG by definition contains a momentum cutoff
  $\longrightarrow$ several generalizations in the market
  $\longrightarrow$ construction of a gauge invariant RG flow equation
  $\longrightarrow$ typically very complicated, hard to use them for practical computations that go beyond usual perturbative treatments
- Goal here: practical computations

Gauge symmetry violation in the FRG

- Abelian gauge (A_i) theory, N scalar fields (ϕ^a)
  $(D_i = \partial_i - ieA_i,\ F_{ij} = \partial_i A_j - \partial_j A_i)$

$$S = \int \left[\frac{1}{4}F_{ij}F_{ij} + (D_i\phi^a)^\dagger D_i\phi^a + m^2\phi^{\dagger a}\phi^a + \frac{\lambda}{6}(\phi^{\dagger a}\phi^a)^2\right] + S_{gf}$$

- Gauge symmetry:

$$\delta\phi^a(x) = ie\Theta(x)\phi^a(x), \quad \delta A_i(x) = -\partial_i\Theta(x)$$

- Quantum theory: gauge symmetry is encoded in the Ward-Takahashi identities $[\Phi^T = (A_i, \phi^{\dagger a}, \phi^a)]$

$$\delta Z[J] = 0 \quad \Rightarrow \quad \int_x \langle\delta\Phi^\alpha(x)\rangle \frac{\delta\Gamma[\bar\Phi]}{\delta\bar\Phi^\alpha(x)} - \langle\delta S\rangle = 0$$

- Covariant gauge fixing: $\delta S = \frac{i}{\xi}\int_p \Theta(-p)p^2 p_i A_i(p)$
  $\rightarrow$ project the master eq. onto different operators: WTIs

Gauge symmetry violation in the FRG

- Projecting onto $\underline{\sim \bar{A}_i}$: WTI of the $A_i A_j$ vertex

$$p_i \Gamma_{ij}(p) = -\frac{1}{\xi} p^2 p_j \;\Rightarrow\; \Gamma_{ij}(p) = (-\delta_{ij} p^2 + p_i p_j)(1 - \Pi(p)) - \frac{1}{\xi} p_i p_j$$

$\longrightarrow$ only the transverse part receives quantum corrections

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- Pr. onto $\underline{\sim \phi^{\dagger a} \phi^b}$: WTI between $\phi^{\dagger a} \phi^b$ and $A_i \phi^{\dagger a} \phi^b$ vertices

$$e\big(\Gamma_{ab}(q+p) - \Gamma_{ab}(q)\big) = p_i \Gamma^i_{ab}(p,q)$$

$\longrightarrow$ scalar and charge rescaling factors agree: $Z_\phi = Z_e$

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- Projecting onto $\underline{\sim \bar{A}_{i_1} \bar{A}_{i_2} \ldots \bar{A}_{i_{n-1}}}$: WTI of the n-point A vertex

$$p_i \Gamma_{i, i_1, i_2, \ldots, i_{n-1}}(p, q_1, q_2, \ldots) = 0$$

Gauge symmetry violation in the FRG

- In the FRG formalism the WTIs change:

$$S \longrightarrow S + \frac{1}{2}\iint \Phi^T R_k \Phi, \quad \Gamma \longrightarrow \Gamma_k + \frac{1}{2}\iint \bar{\Phi}^T R_k \bar{\Phi}$$

- The modified Ward-Takahashi identities (mWTI) are generated via

$$\int_x \langle \delta\Phi^\alpha(x)\rangle \frac{\delta\Gamma[\bar{\Phi}]}{\delta\bar{\Phi}^\alpha(x)} - \langle \delta S\rangle = \langle \delta \iint \frac{1}{2}\Phi^T R_k \Phi\rangle - \int \langle \delta\Phi^\alpha\rangle \frac{\delta}{\delta\bar{\Phi}^\alpha} \iint \frac{1}{2}\bar{\Phi}^T R_k \bar{\Phi}$$

- The gauge field does not contribute to the rhs:

$$\int_x \langle \delta\Phi^\alpha(x)\rangle \frac{\delta\Gamma_k[\bar{\Phi}]}{\delta\bar{\Phi}^\alpha(x)} - \langle \delta S\rangle = ie\int_p \Theta(-p)\int_q \langle \phi_a^\dagger(q)\phi_a(q+p)\rangle_c \times [R_k(p+q) - R_k(q)]$$

Applications: Abelian gauge theory

- LPA ansatz for the effective action with optimal regulator: $[R_k(q) = (k^2 - q^2)\Theta(k^2 - q^2)]$

$$\Gamma_k = \int \Bigg[\frac{Z_{A,k}}{2} A_i[-\partial^2\delta_{ij} + \partial_i\partial_j(1-\xi_k^{-1})]A_j + \frac{1}{2}m^2_{A,k}A_iA_i$$
$$+ Z_{\phi,k}\partial_i\phi^{\dagger a}\partial_i\phi^a + iZ_{e,k}eA_i(\phi^\dagger_a\partial_i\phi_a - \partial_i\phi^\dagger_a\phi_a)$$
$$+ \frac{Z^2_{e,k}}{Z_{\phi,k}}e^2A_iA_i\phi^\dagger_a\phi_a + \frac{Z_{\phi,k}m_k^2}{2}\phi^{\dagger a}\phi^a + \frac{Z^2_{\phi,k}\lambda_k}{6}(\phi^{\dagger a}\phi^a)^2\Bigg]$$

- Proposal: try to maintain all WTIs related to the ansatz (1) $Z_{e,k} = Z_{\phi,k}$ (2) transverse corrections to the gauge prop.
- The $\sim \phi^{\dagger a}\phi^b$ projection leads to

$$\frac{Z_{e,k}}{Z_{\phi,k}} = 1 + 2e_k^2\xi_k \int_q \frac{R_k(q)}{(q^2 + R_k(q))^2}$$

$\longrightarrow$ choose the Landau gauge ($\xi = 0$)!

Applications: Abelian gauge theory

- The $\sim A_i$ projection leads to two conditions $[\mathcal{O}(p^0), \mathcal{O}(p^2)]$

$$m^2_{A,k} = -\frac{-4Ne^2}{d(d+2)}\Omega_d k^{d-2}, \quad \frac{Z_{A,k}}{\xi_k} - \frac{1}{\xi_\Lambda} = \Omega_d e^2 \frac{4k^{d-4}}{d(d+2)}$$

$\longrightarrow$ photon mass is irrelevant $(m^2_{A,k} \to 0$, if $k \to 0)$
$\longrightarrow$ plugging in the flow of $Z_{A,k}$ one gets $\xi_k \equiv 2/(d-4) \neq 0$!
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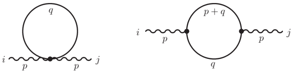
- Exactly the same can be obtained from the flow equation:

$$k\partial_k \Gamma_{ij}(p) = \int k\tilde{\partial}_k \Bigg[\qquad \qquad \Bigg]$$



$\longrightarrow$ photon mass flows but dies out in the IR
$\longrightarrow$ flow of the longitudinal propagator is nonzero
$\Rightarrow$ it can be compensated with the above choice of ξ_k

Applications: Abelian gauge theory

- What to do with the discrepancy of ξ?
  $\longrightarrow$ in $d = 4$ there is no problem (pert. results are recovered)

$$k\partial_k \lambda_k = \frac{54e_k^4 - 18e_k^2\lambda_k + (N+4)\lambda_k^2}{24\pi^2}, \quad k\partial_k e_k^2 = \frac{Ne_k^4}{24\pi^2}$$

- In $d \neq 4$ one way is to enforce $Z_{\phi,k} = Z_{e,k}$ by hand and choose $\xi = 2/(4-d)$ at the same time
  $\longrightarrow$ ξ enters the β functions!

[1] GF & T. Hatsuda, Phys. Rev. D**93**, 121701 (2016)
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- In $d\neq 4$ one way is to enforce $Z_{\phi,k}=Z_{e,k}$ by hand and choose $\xi=2/(4-d)$ at the same time
  $\longrightarrow$ ξ enters the β functions!
- Example of superconductivity: Abelian Higgs model with $N=1$ complex scalar
  $\longrightarrow$ perturbative results do not allow for an IR fixed point, but it does exist (predictions of Monte-Carlo simulations)
  $\longrightarrow$ the above method does produce an IR fixed point[1]

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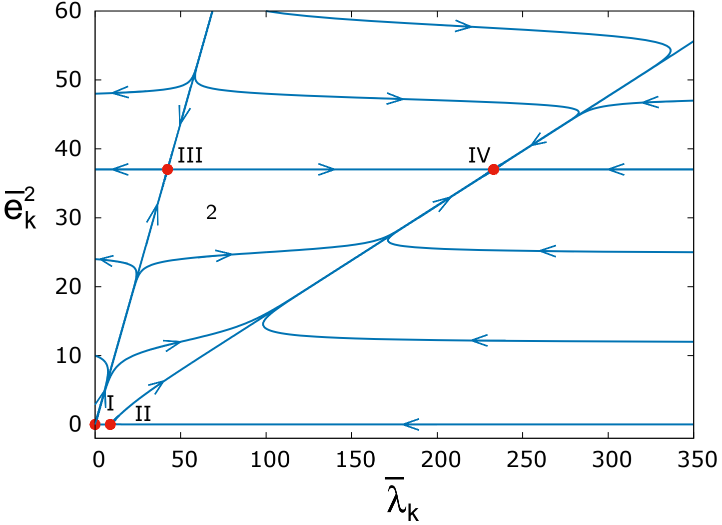

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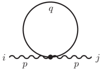
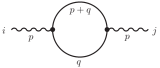
Applications: Abelian gauge theory

- Can we save the transversality of the gauge propagator?

$$k\partial_k \Gamma_{ij}(p) = k\tilde{\partial}_k \Big[$$





$$\Big]\Big|_{\mathcal{O}(p^2)}$$

$$= -e^2 \int_q k\partial_k \frac{(p+2q)_i}{q^2 + R_k(q)} \frac{(p+2q)_j}{(q+p)^2 + R_k(q+p)}\Bigg|_{\mathcal{O}(p^2)}$$

$$\sim p^2\delta_{ij} - \left(1 - \frac{4-d}{2}\right) p_i p_j$$

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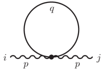

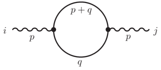

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$$\sim p^2\delta_{ij} - \left(1-\frac{4-d}{2}\right)p_ip_j$$

- We may try to regulate the vertex momenta!

$$\rightarrow \vec{q}_{R_k^\alpha} = \vec{q} + \alpha(k\vec{\hat{q}} - \vec{q})\Theta(k^2-q^2)$$

$$k\partial_k\Gamma_{ij}(p) = -e^2\int_q k\partial_k\frac{(p+q)_{i,R_k^\alpha}+q_{i,R_k^\alpha}}{q^2+R_k(q)}\frac{(p+q)_{j,R_k^\alpha}+q_{j,R_k^\alpha}}{(q+p)^2+R_k(q+p)}\Big|_{\mathcal{O}(p^2)}$$

$$\sim \underline{\underline{p^2\delta_{ij} - p_ip_j}}$$

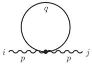
if $\alpha^2 = 2(2-d)$!

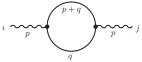
Applications: Abelian gauge theory

- Is it justified to regulate the vertex momenta? $\rightarrow$ yes!
- Momentum dependence of the vertex comes from the derivative coupling: $[\phi^a = (s^a + i\pi^a)/\sqrt{2}]$

$$e \int_x A_i (s^a \partial_i \pi^a - \pi^a \partial_i s^a)$$

$$\rightarrow ie \int A_i(q_1) s^a(q_2) \pi^a(q_3)(q_3 - q_2)_i \delta(q_1 + q_2 + q_3)$$

- The 



diagrams are generated when A_i is set to a background field:

$$\rightarrow ie \int s^a(q_2) \pi^a(q_3)(q_{3i} - q_{2i}) \bar{A}_i(-q_2 - q_3)$$

- A non-diagonal, background dependent regulator is needed!

$$\rightarrow ie \int s^a(q_2) \pi^a(q_3)(q_{3i, R_k^\alpha} - q_{2i, R_k^\alpha}) \bar{A}_i(-q_2 - q_3)$$

Applications: non-Abelian gauge theory

- Gauge symmetry: [Non-Abelian Higgs model, ϕ has flavor and color]

$$\delta\phi_\gamma(x) = ig\Theta^a\hat{T}^a\phi_\gamma(x), \quad \delta A_i^a(x) = -\partial_i\Theta^a(x) + gf^{abc}A_i^b\Theta^c$$

- LPA ansatz for the effective action:

$$\begin{aligned}
\Gamma_k \quad = \quad & \int_x \Bigg[\frac{Z_{A,k}}{2}A_i^a\delta^{ab}\Big(-\partial^2\delta_{ij} + (1-\xi_k^{-1})\partial_i\partial_j\Big)A_j^b \\
& + Z_{c,k}\bar{c}^a(-\partial^2\delta^{ac} - Z_{g,k}Z_{A,k}^{1/2}gf^{abc}\partial_i A_i^b)c^c \\
& - Z_{\phi,k}\phi_\gamma^{\dagger n}\partial^2\phi_\gamma^n + \frac{Z_{\phi,k}m_k^2}{2}\phi_\gamma^{\dagger n}\phi_\gamma^n + \frac{Z_{\phi,k}^2\lambda_k}{6}(\phi_\gamma^{\dagger n}\phi_\gamma^n)^2 \\
& + \mathrm{i}Z_{g,k}Z_{A,k}^{1/2}Z_{\phi,k}gA_i^a\Big(\partial_i\phi_\gamma^\dagger(\hat{T}^a\phi_\gamma) - (\hat{T}^a\phi_\gamma)^\dagger\partial_i\phi_\gamma\Big) \\
& + Z_{g,k}^2 Z_{A,k}Z_{\phi,k}g^2 f^{abe}f^{cde}A_i^aA_i^b(\hat{T}^c\phi_\gamma)^\dagger(\hat{T}^d\phi_\gamma) \\
& + Z_{g,k}Z_{A,k}^{3/2}gf^{abc}\partial_i A_j^a A_i^b A_j^c \\
& + Z_{g,k}^2 Z_{A,k}^2\frac{g^2}{4}f^{abe}f^{cde}A_i^aA_j^bA_i^cA_j^d\Bigg]
\end{aligned}$$

Applications: non-Abelian gauge theory

- Generalized WTIs: (Slavnov-Taylor identities)
  $\longrightarrow$ formally same as the Abelian but much more complicated

$$\int_x \langle \delta \Phi^\alpha(x) \rangle \frac{\delta \Gamma[\bar{\Phi}]}{\delta \bar{\Phi}^\alpha(x)} - \langle \delta S \rangle = 0$$

- Examples:
  $\longrightarrow$ $Z_{\phi^2 A} - Z_\phi$ and $Z_{\phi^2 A^2} - Z_\phi$ are matter independent (but not zero!)
  $\longrightarrow$ identical definitions of β function of the gauge coupling

$$\begin{aligned}
&- g\mu\partial_\mu(\delta Z_{\phi^2 A} - \delta Z_\phi - \delta Z_A/2) \\
&- g\mu\partial_\mu(\delta Z_{\phi^2 A^2} - \delta Z_\phi - \delta Z_A)/2 \\
&- g\mu\partial_\mu(\delta Z_{3g} - 3\delta Z_A/2) \\
&- g\mu\partial_\mu(\delta Z_{4g}/2 - \delta Z_A) \\
&- g\mu\partial_\mu(\delta_{A\bar{c}c} - \delta Z_c - \delta Z_A/2)
\end{aligned}$$

- These are all broken due to the regulator!

Applications: non-Abelian gauge theory

- Suggestion: to keep Γ_k gauge invariant, deal with all the Slavnov-Taylor identities relevant in the ansatz of Γ_k!

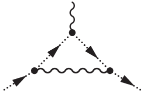
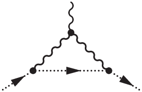
Applications: non-Abelian gauge theory

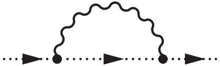
- Suggestion: to keep Γ_k gauge invariant, deal with all the Slavnov-Taylor identities relevant in the ansatz of Γ_k!
- Flow of gauge coupling: e.g. use the definition via the gauge-ghost-ghost vertex (does not include matter!)
- Formally:

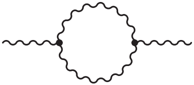
$$\begin{aligned}
\beta(g) &\rightarrow g_k k\partial_k \log Z_{g,k} \\
&= \frac{g_k}{Z_{g,k}} k\partial_k \left(\frac{Z_{g,k} Z_{A,k}^{1/2} Z_{c,k}}{Z_{A,k}^{1/2} Z_{c,k}} \right) \\
&= \frac{g_k}{Z_{g,k} Z_{A,k}^{1/2} Z_{c,k}} k\partial_k (Z_{g,k} Z_{A,k}^{1/2} Z_{c,k}) \\
&\quad - \frac{g_k}{Z_{c,k}} k\partial_k Z_{c,k} - \frac{g_k}{2Z_{A,k}} k\partial_k Z_{A,k}
\end{aligned}$$

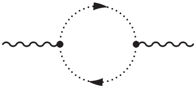
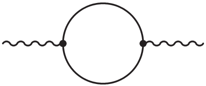
- Use diagrammatics to calculate the red terms!

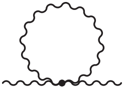
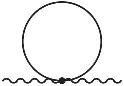
- The following diagrams need to be calculated:[3]

$$p_i g f^{abc} k\partial_k\left(Z_{g,k} Z_{A,k}^{1/2} Z_{c,k}\right) = k\tilde{\partial}_k \Bigg($$  $$+$$  $$\Bigg)$$

$$p^2\delta^{ab} k\partial_k Z_{c,k} = k\tilde{\partial}_k \Bigg($$  $$\Bigg)$$

$$k\partial_k\left[Z_{A,k}\left(p^2\delta_{ij} - p_i p_j(1-\xi_k^{-1})\right)\right]\delta^{ab} = k\tilde{\partial}_k \Bigg($$ 

$$+$$  $$+$$ 

$$+$$  $$+$$  $$\Bigg)$$

[3] GF & N. Yamamoto, JHEP **12** (2019) 069

Applications: non-Abelian gauge theory

- Once again gauge choice is in general not arbitrary!

$$\partial_k \Big[Z_{A,k} \big(p^2 \delta_{ij} - p_i p_j (1 - \xi_k^{-1}) \big) \Big] = (\dots) \big[p^2 \delta_{ij} - p_i p_j f(d) \big]$$

- For $d = 4$ the flow is transverse and we get

$$\beta_g|_{d=4} = -\frac{g_k^3}{(4\pi)^2} \Big(\frac{11}{3} N_c - \frac{N_f}{6} \Big)$$

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- For $d = 3$ the flow is not transverse $\rightarrow$ two roots for ξ!

$$\xi_\pm = 1 + \frac{N_\mathrm{f}}{4} \pm \frac{1}{4} \sqrt{N_\mathrm{f}^2 - 8 N_\mathrm{f} + 456}$$

- Requirement: since at large N_f gauge fields are suppressed, there should be an IR fixed point $\rightarrow$ rules out ξ_+

Applications: non-Abelian gauge theory

- The β-function in $d = 3$:

$$\begin{aligned} \beta(g)|_{d=3} &= -g_k - \frac{g_k^3}{2\pi^2}\left[\left(\frac{19}{9} + \frac{16}{45}\xi_-\right)N_\mathrm{c} - \frac{2N_\mathrm{f}}{15}\right] \\ \xi_- &= 1 + \frac{N_\mathrm{f}}{4} - \frac{1}{4}\sqrt{N_\mathrm{f}^2 - 8N_\mathrm{f} + 456} \end{aligned}$$

- Application: color superconductivity ($N_c = 3,\ N_f = 3$)
  $\longrightarrow$ $\beta(g) < 0$ for all $g > 0$
  $\longrightarrow$ no IR fixed point, regardless of the scalar potential!
  $\longrightarrow$ color superconducting phase transition is of 1st order
- Can the flow of the gluon propagator be transverse? Appropriate vertex regularization?
  $\Longrightarrow$ future work!

Summary

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- Local Potential Approx. + Abelian gauge theories $\rightarrow$ OK
- Local Potential Approx. + non-Abelian gauge theories
  $\longrightarrow$ transversality of the gluon propagator?
  $\longrightarrow$ equivalence of choices for the flowing gauge coupling?