Functional Renormalization Group Analysis of Driven Disordered Systems

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Outline

1. Introduction: FRG formalism for disordered systems
2. Model: driven random manifold (DRM)
3. Zero-temperature case
    - Dimensional reduction
    - Breakdown of dimensional reduction in DRM
    - BKT transition in three-dimensional random field driven XY model
4. Finite-temperature case
    - Relevance of temperature in nonequilibrium stead states
    - Anomalous crossover from finite- to zero-temperature cases
5. Summary

Introduction

FRG formalism for disordered systems

Competition between interactions and fluctuations

Interaction
$V_{ij} = -JS_i \cdot S_j$
System ⇒ Ordered

Competition

Thermal noise
$\xi(r, t)$
System ⇒ Disordered

Phase transitions and critical phenomena in pure systems
Crystals, Ferromagnets, BEC, Superconductors, Liquid crystals,…

Interaction
$V_{ij} = -JS_i \cdot S_j$
System ⇒ Ordered

Competition

Quenched disorder
$h(r)$
System ⇒ More Disordered

Phase transitions and critical phenomena in disordered systems

Field theoretical formalism for disordered systems

- ✓ Hamiltonian of a disordered system: $H[\phi; h]$ ($h(r)$: disorder)
- ✓ Partition function and free energy

$$Z[J; h] := \int \mathcal{D}\phi \exp(-H[\phi; h] + J \cdot \phi)$$

$J \cdot \phi := \int J(r)\phi(r)dr$

$$\Rightarrow \; W[J; h] := \ln Z[J; h]$$

- ✓ Connected and disconnected Green functions

$$G_c(r_1, r_2) := \overline{\langle\phi(r_1)\phi(r_2)\rangle_h - \langle\phi(r_1)\rangle_h\langle\phi(r_2)\rangle_h} = \overline{\frac{\delta^2 W[J; h]}{\delta J(r_1)\delta J(r_2)}}$$

$$G_d(r_1, r_2) := \overline{\langle\phi(r_1)\rangle_h\langle\phi(r_2)\rangle_h} - \overline{\langle\phi(r_1)\rangle_h}\;\overline{\langle\phi(r_2)\rangle_h}$$

$$= \overline{\frac{\delta W[J; h]}{\delta J(r_1)}\frac{\delta W[J; h]}{\delta J(r_2)}} - \overline{\frac{\delta W[J; h]}{\delta J(r_1)}}\;\overline{\frac{\delta W[J; h]}{\delta J(r_1)}}$$

What we want to calculate

Replica field theory

It is difficult to calculate $\overline{W[J;h]} = \overline{\ln Z[J;h]}$.

⇒ Introduce n replicas of the system with the same disorder

✓ Replicated fields and sources: $\{\phi_a\}_{a=1,\dots,n}$ and $\{J_a\}_{a=1,\dots,n}$

✓ Replicated partition function and free energy

$$Z[\{J_a\}] \coloneqq \overline{\prod_{a=1}^{n} Z[J_a;h]} = \int \mathcal{D}\phi_a \, \overline{\exp\left(-\sum_a H[\phi_a;h] + J_a \cdot \phi_a\right)}$$

$$\Rightarrow \; W[\{J_a\}] \coloneqq \ln Z[\{J_a\}]$$

This disorder average can be easily calculated.

✓ Replicated effective action

$$\Gamma[\{\psi_a\}] = -W[\{J_a\}] + J_a \cdot \psi_a, \qquad \psi_a = \langle \phi_a \rangle = \frac{\delta W[\{J_a\}]}{\delta J_a}$$

Replica field theory

✓ Cumulant expansion of effective action

$$\Gamma[\{\psi_a\}] = \sum_a \Gamma_1[\psi_a] - \frac{1}{2}\sum_{a,b}\Gamma_2[\psi_a,\psi_b] + \cdots$$

$$\Gamma_1[\psi_a] = \overline{\Gamma[\psi_a]}$$
$$\Gamma_2[\psi_a,\psi_b] = \overline{\Gamma[\psi_a]\Gamma[\psi_b]} - \overline{\Gamma[\psi_a]}\ \overline{\Gamma[\psi_b]}$$

✓ Connected and disconnected Green functions

$$G_c(r_1,r_2\,) = \left(\frac{\delta^2\Gamma_1[\psi]}{\delta\psi\delta\psi}\right)^{-1}_{r_1,r_2}$$

$$G_d(r_1,r_2) = \int_{r_3,r_4} G_c(r_1,r_3)\frac{\delta^2\Gamma_2[\psi_a,\psi_b]}{\delta\psi_a(r_3)\delta\psi_b(r_4)}G_c(r_4,r_2)$$

See [G. Tarjus and M. Tissier, PRB 78 024203 (2008)] for details

What we need to calculate is $\Gamma_1[\psi_a]$ and $\Gamma_2[\psi_a,\psi_b]$.

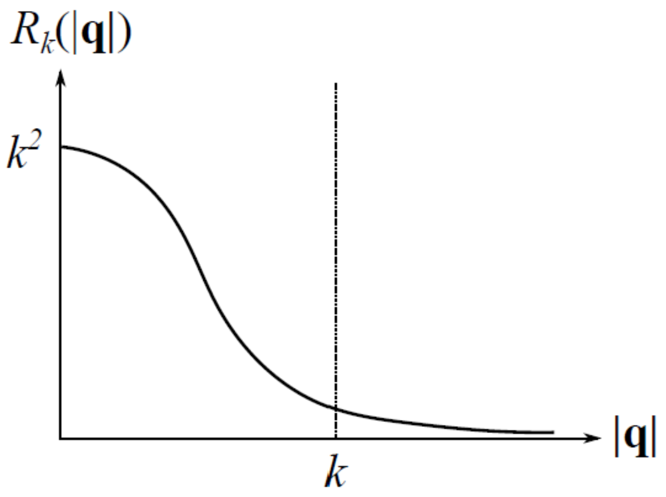
FRG formalism with replica

✓ Scale-dependent mass term

$$\Delta H_k[\{\phi_a\}] = \frac{1}{2}\sum_a \int dq\, R_k(q)\phi_a(q)\phi_a(-q)$$

$$\Rightarrow Z_k[\{J_a\}] = \int \mathcal{D}\phi_a \exp(\cdots - \Delta H_k[\{\phi_a\}])$$

$$\Rightarrow \Gamma_k[\{\psi_a\}] = -\ln Z_k[\{J_a\}] + J_a \cdot \psi_a - \Delta H_k[\{\psi_a\}]$$



✓ Exact flow equation for $\Gamma_k[\{\psi_a\}]$

$$\partial_k \Gamma_k[\{\psi_a\}] = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k \left[\frac{\partial^2 \Gamma_k[\{\psi_a\}]}{\partial\psi_a \partial\psi_b} + R_k\right]^{-1}\right)$$

Note: $[\cdots]^{-1}$ involves the inversion of $n \times n$ matrix for the replica index.

FRG formalism with replica

✓ Cumulant expansion of scale-dependent effective action

$$\Gamma_k[\{\psi_a\}] = \sum_a \Gamma_{1,k}[\psi_a] - \frac{1}{2}\sum_{a,b} \Gamma_{2,k}[\psi_a, \psi_b] + \frac{1}{3!}\sum_{a,b,c} \Gamma_{3,k}[\psi_a, \psi_b, \psi_c] - \cdots$$

✓ Hierarchy of exact flow equations

$$\partial_k \Gamma_{1,k}[\psi] = (\text{derivatives of } \Gamma_{1,k}[\psi], \Gamma_{2,k}[\psi_a, \psi_b])$$

$$\partial_k \Gamma_{2,k}[\psi_a, \psi_b] = (\text{derivatives of } \Gamma_{1,k}[\psi], \Gamma_{2,k}[\psi_a, \psi_b], \Gamma_{3,k}[\psi_a, \psi_b, \psi_c])$$

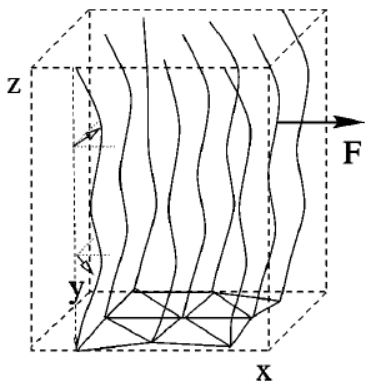
$\cdots$

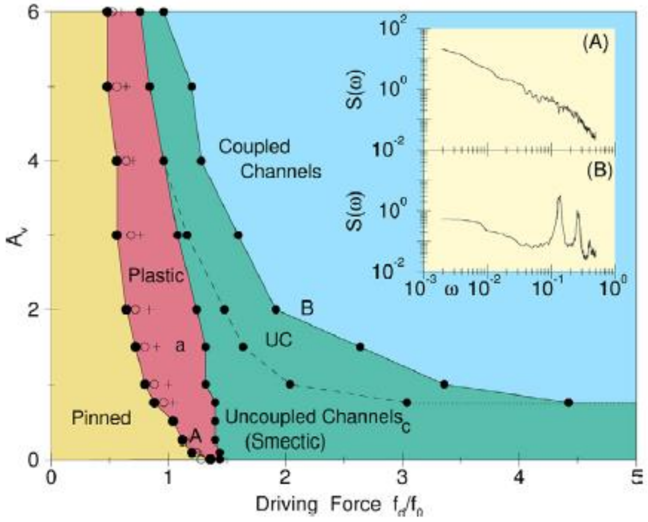
Note: To truncate the hierarchy, it is often assumed that $\Gamma_{3,k} = \Gamma_{4,k} = \cdots = 0$.

See [G. Tarjus and M. Tissier, PRB 78 024203 (2008)] for details

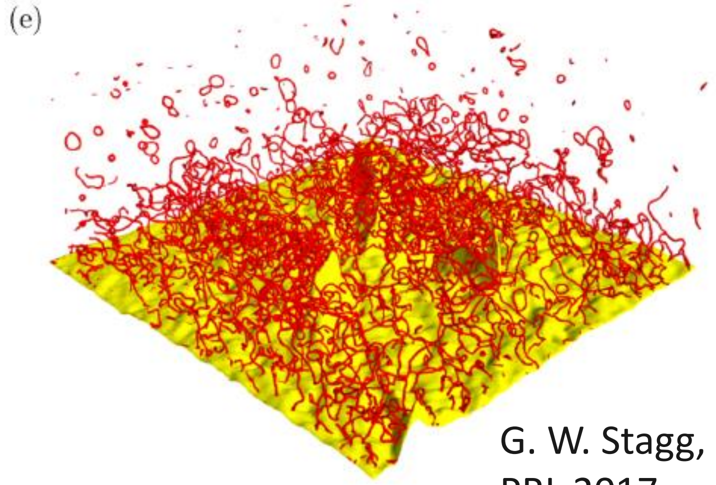
Nonequilibrium systems

✓ What happens when a disordered system is driven out of equilibrium?





C. J. Reichhardt, et. al., PRL 1998



G. W. Stagg, et. al. PRL 2017

Driven vortex lattices in dirty superconductors

Superfluid flowing in random media

Too complicated for theoretical analysis ⇒ Introduction of a toy model

Model

Driven random manifold

Elastic manifold in random potential at equilibrium

✓ Displacement field: $\phi = (\phi^1, \dots, \phi^N)$

✓ Hamiltonian:

$$H_{\mathrm{RM}}[\phi; V] = \int dr \left[\frac{1}{2} |\nabla \phi(r)|^2 + V\big(r; \phi(r)\big) \right]$$

Elastic energy

Random potential

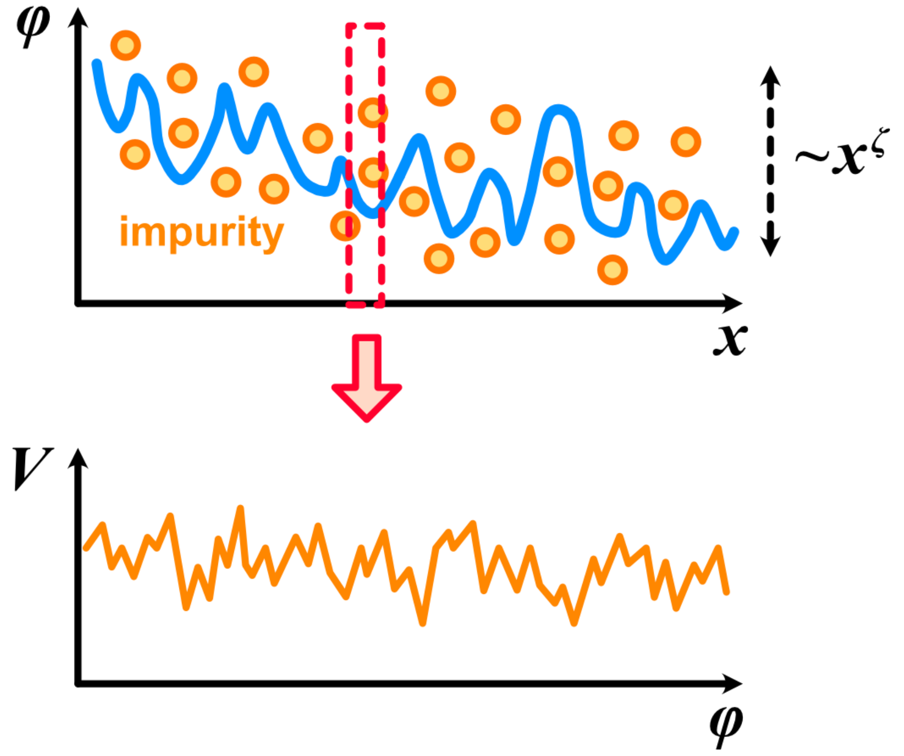
✓ Random force:

$$F^\alpha(r;\phi) \coloneqq -\partial_{\phi^\alpha} V(r;\phi)$$

$$\begin{cases} \overline{F^\alpha(r;\phi)} = 0 \\ \overline{F^\alpha(r;\phi) F^\beta(r';\phi')} = \Delta^{\alpha\beta}(\phi - \phi')\delta(r - r') \end{cases}$$

Disorder correlator

$\Delta^{\alpha\beta}(\phi)$: short-range function ($\sim e^{-O(\phi)}$)



P. Le Doussal et al., PRE 69, 026112 (2004)

Elastic manifold in random potential at equilibrium

✓ Mean-square displacement:

$$B(r-r') \coloneqq \overline{\langle[\phi(r)-\phi(r')]^2\rangle} \sim |r-r'|^{2\zeta_{\mathrm{eq}}}$$

$\Rightarrow$ ζ_{eq}: Roughness exponent

✓ Trivial case: $F^{\alpha}(r;\phi)$ is independent of ϕ

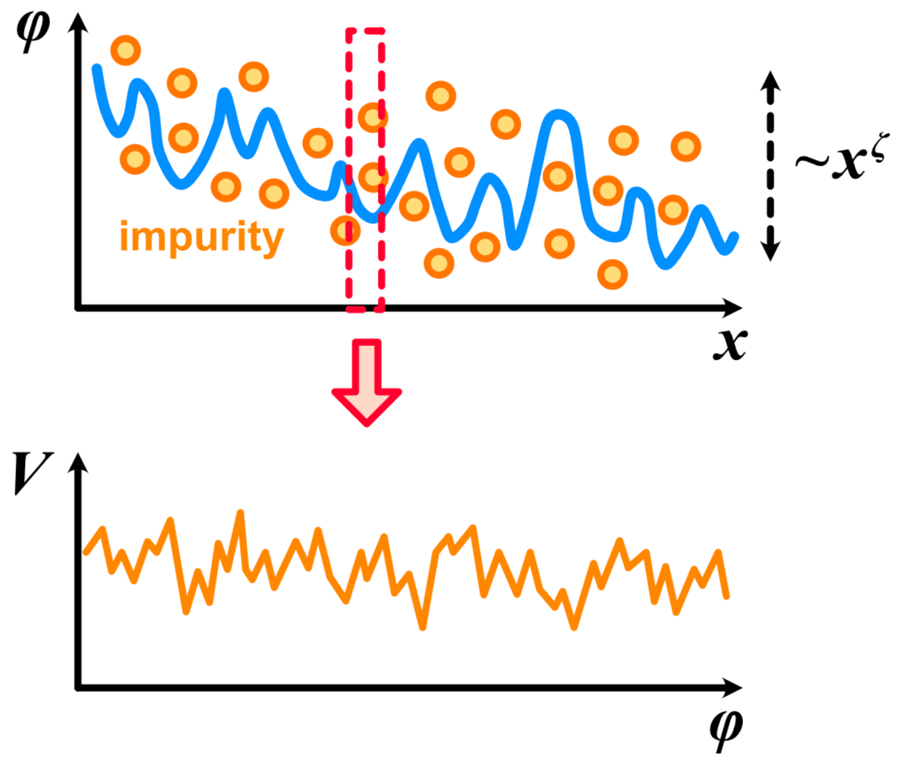
$$\Rightarrow \zeta_{\mathrm{eq}} = \frac{4-D}{2} \quad (D \leq 4)$$

✓ Short-range disorder ($N=1$)

$\zeta_{\mathrm{eq}} = 2/3 \ (D=1\text{: line})$ — Exact

$\zeta_{\mathrm{eq}} \simeq 0.41 \ (D=2\text{: membrane})$

$\zeta_{\mathrm{eq}} \simeq 0.22 \ (D=3\text{: bulk})$ — Simulation



P. Le Doussal et al., PRE 69, 026112 (2004)

Dimensional reduction and its failure

✓ Dimensional reduction

Naïve perturbation theory predicts the equivalence between

D-dim. disordered system & $(D-2)$-dim. system without disorder

A. P. Young, J. Phys. C 10, L257 (1977); G. Parisi and N. Sourlas, PRL 43 744 (1979)

⇩

Roughness exponent: $\zeta_{\mathrm{eq,DR}} = \dfrac{4-D}{2}$ — Same as the trivial case!

⇒ Contradicts experimental and numerical values of ζ_{eq}

Naïve perturbation theory fails to capture a complex energy landscape.

M. Mézard and G. Parisi, J. Phys. I (Paris) 1, 809 (1991)

Driven random manifold

✓ Langevin dynamics:

Thermal noise

$$\partial_t \phi = -\frac{\delta H_{\mathrm{RM}}[\phi; V]}{\delta \phi} + F_{\mathrm{NP}}[\phi] + \xi$$

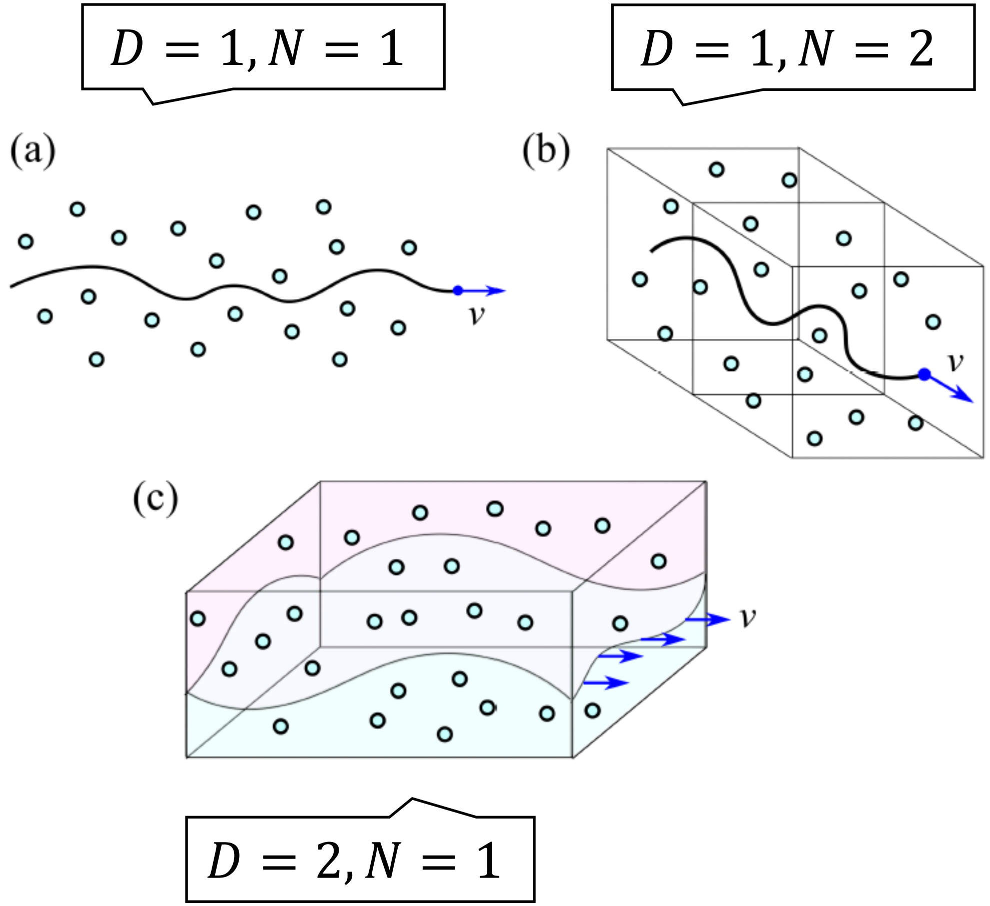
✓ Conditions for $F_{\mathrm{NP}}[\phi]$:

1. Non-potential: $F_{\mathrm{NP}}[\phi] \neq -\delta U[\phi]/\delta\phi$
2. O(N) symmetry: $F_{\mathrm{NP}}[\mathcal{R}(\phi)] = \mathcal{R}(F_{\mathrm{NP}}[\phi])$
3. Linearity: $F_{\mathrm{NP}}[c\phi] = cF_{\mathrm{NP}}[\phi]$

Rotation

✓ Simplest choice:

$F_{\mathrm{NP}}[\phi] = -v\partial_x \phi$ （convection term）



Results

Zero-temperature cases

T. H., J. Stat. Mech. (2019) 073301

T. H., Phys. Rev. B 96, 184202 (2017)

T. H., Phys. Rev. E 98, 032122 (2018)

Dimensional reduction for driven disordered system

Steady state equation: $v\partial_x\phi^\alpha = \nabla^2\phi^\alpha + F^\alpha(r;\phi)$

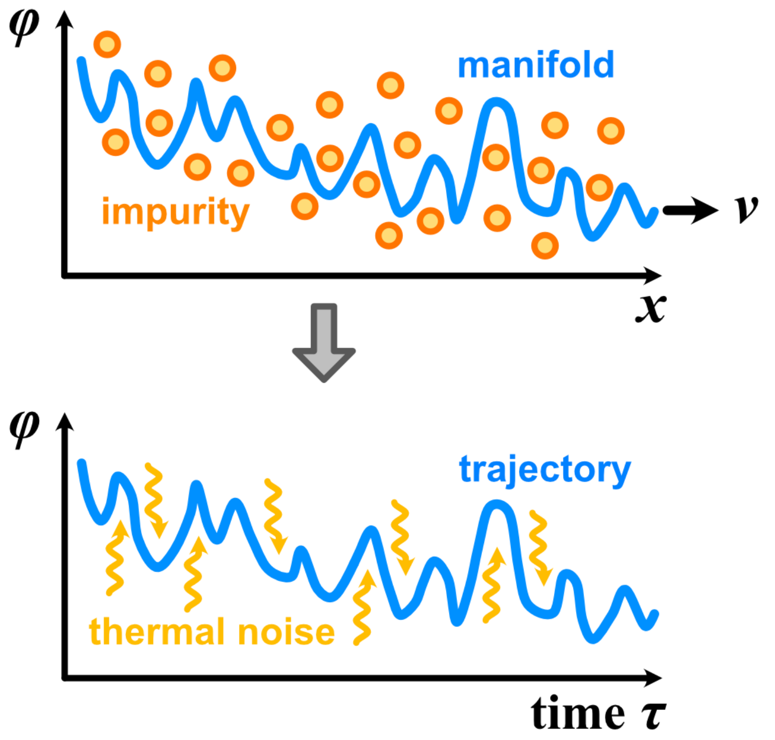
- $\partial_x^2\phi^\alpha \ll v\partial_x\phi^\alpha$ in the large-scale

Omitting $\partial_x^2\phi^\alpha \Rightarrow v\partial_x\phi^\alpha = \nabla_\perp^2\phi^\alpha + F^\alpha(r;\phi)$

- spatial coordinate $x \to$ "time" τ
- random force $F^\alpha \to$ "thermal noise" η^α

$$v\partial_\tau\phi^\alpha(\tau,r_\perp) = \nabla_\perp^2\phi^\alpha(\tau,r_\perp) + \eta^\alpha(\tau,r_\perp)$$

$$\langle\eta^\alpha(\tau,r_\perp)\eta^\beta(\tau',r'_\perp)\rangle = \Delta(0)\delta(\tau-\tau')\delta(r_\perp - r'_\perp)$$



$\Rightarrow$ Dynamics of $(D-1)$-dim. manifold with temperature $T_{\mathrm{eff}} = \Delta(0)/2v$

$\Rightarrow$ Roughness exponent: $\zeta_{\perp,\mathrm{DR}} = (3-D)/2$, $\zeta_{\parallel,\mathrm{DR}} = (3-D)/4$

Breakdown of the dimensional reduction

✓ Bare disorder correlator:

$$\overline{F^{\alpha}(r;\phi)F^{\beta}(r';\phi')} = \Delta^{\alpha\beta}(\phi-\phi')\delta(r-r')$$

Integrating out modes with $e^{-l}\Lambda < q < \Lambda$

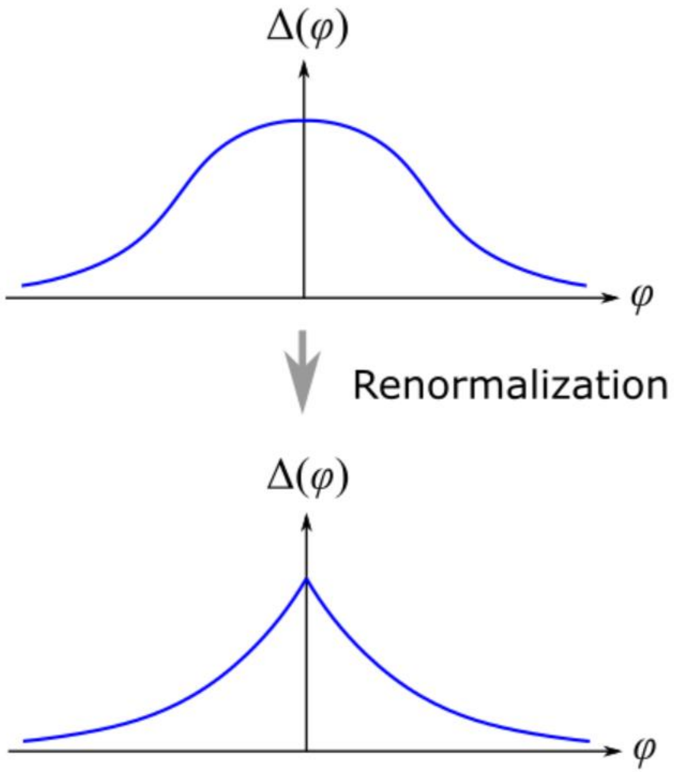
✓ Renormalized disorder correlator:

$$\overline{\tilde{F}^{\alpha}(r;\phi)\tilde{F}^{\beta}(r';\phi')} = \tilde{\Delta}_{l}^{\alpha\beta}(\phi-\phi')\delta(r-r')$$

$l = -\log k$

✓ Fixed point: $\partial_l \tilde{\Delta}_{*}^{\alpha\beta}(\phi) = 0$

- $\tilde{\Delta}_{*}^{\alpha\beta}(\phi)$ is analytic at $\phi = 0$. $\Rightarrow$ The DR holds.
- $\tilde{\Delta}_{*}^{\alpha\beta}(\phi)$ has a cusp at $\phi = 0$. $\Rightarrow$ The DR can fail.

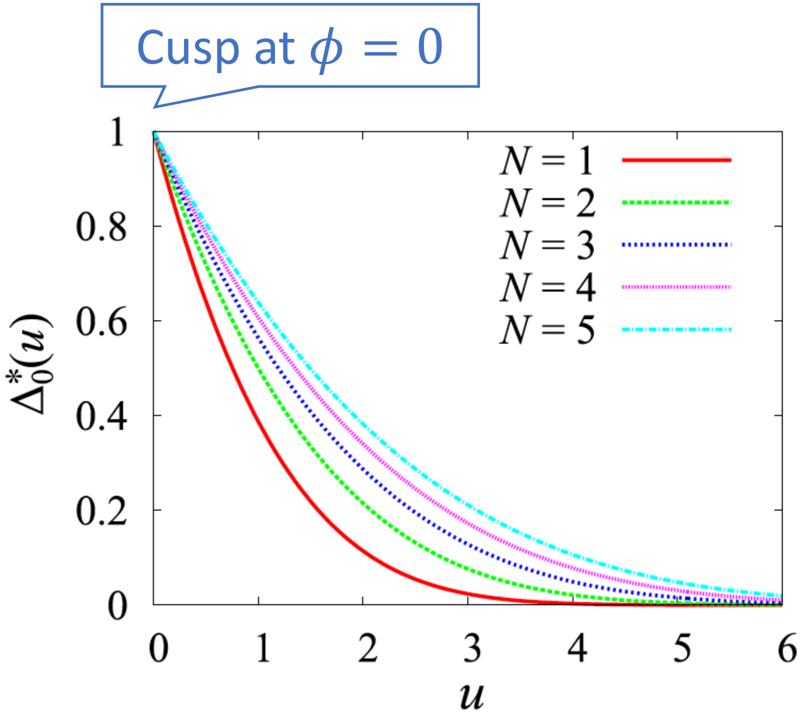


Non-analyticity of disorder correlator $\Rightarrow$ Metastability of zero-temp. states

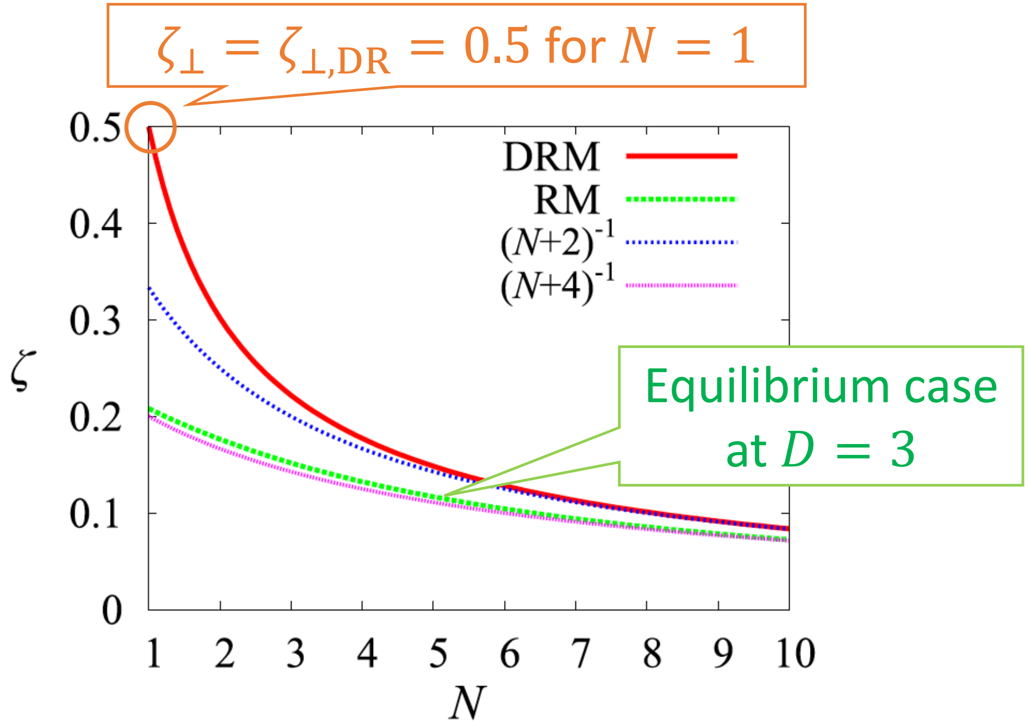
Equilibrium cases: L. Balents et al., J. Phys. I (Paris) 6 1007 (1996)

Renormalized disorder correlator

✓ Fixed point $\partial_l \tilde{\Delta}_*^{\alpha\beta}(\phi) = 0$ at $D = 2$



✓ Roughness exponent $\zeta_\perp$ at $D = 2$



- The breakdown of the DR is weaker than the equilibrium case.
- The DR recovers for $N = 1$, despite of a cuspy behavior of $\tilde{\Delta}_*^{\alpha\beta}(\phi)$.

Driven random field XY model

✓ Two-component field: $\Phi = (\Phi^1, \Phi^2)$

✓ Hamiltonian:

$$H[\Phi] = \int dr \left[\frac{1}{2}|\nabla\Phi|^2 + U(|\Phi|) - h \cdot \Phi\right]$$

✓ Potential: $U(|\Phi|) = \frac{\kappa}{2}(|\Phi| - 1)^2$

✓ Random field: $h(r) = \left(h^1(r), h^2(r)\right)$

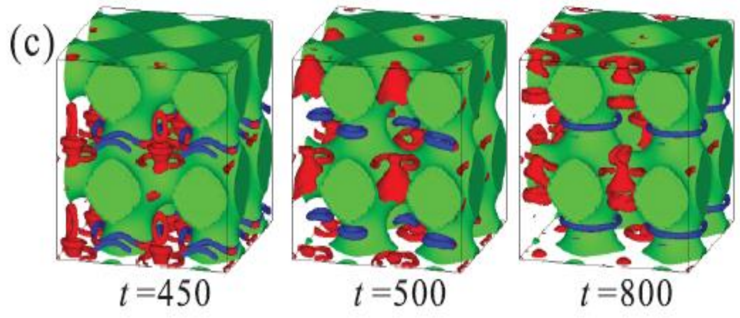
$\Rightarrow \overline{h^\alpha(r) h^\beta(r')} = h_0^2 \delta^{\alpha\beta} \delta(r - r')$

✓ Dynamics:

$$\partial_t \Phi^\alpha + v \partial_x \Phi^\alpha = -\frac{\delta H[\Phi]}{\delta \Phi^\alpha} + \xi^\alpha$$

Example: Liquid crystal flowing in a porous medium

- Irregular surface of a porous medium ⇒ Disorder
- Flow sustained by a pressure gradient ⇒ Driving



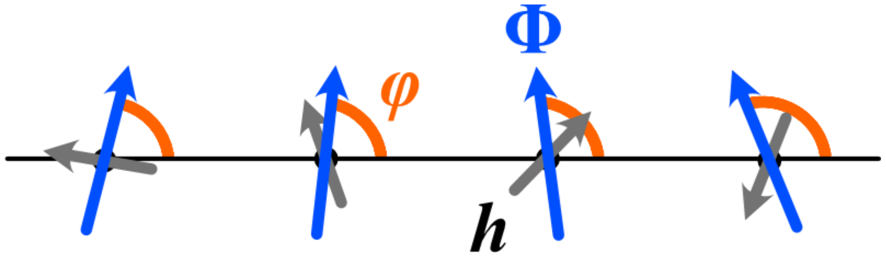
T. Araki, PRL 109, 257801 (2012)

Spin-wave approximation of DRFXYM

✓ Phase variable: $\Phi = (\Phi^1, \Phi^2) = (\cos\phi\,, \sin\phi)$

✓ Hamiltonian:

$$H[\phi] = \int dr \left[\frac{1}{2}|\nabla\phi|^2 - h^1\cos\phi - h^2\sin\phi\right]$$



✓ Dynamics:

$$\partial_t\phi(r) + v\partial_x\phi(r) = \nabla^2\phi(r) + F(r;\phi) + \xi(r,t)$$

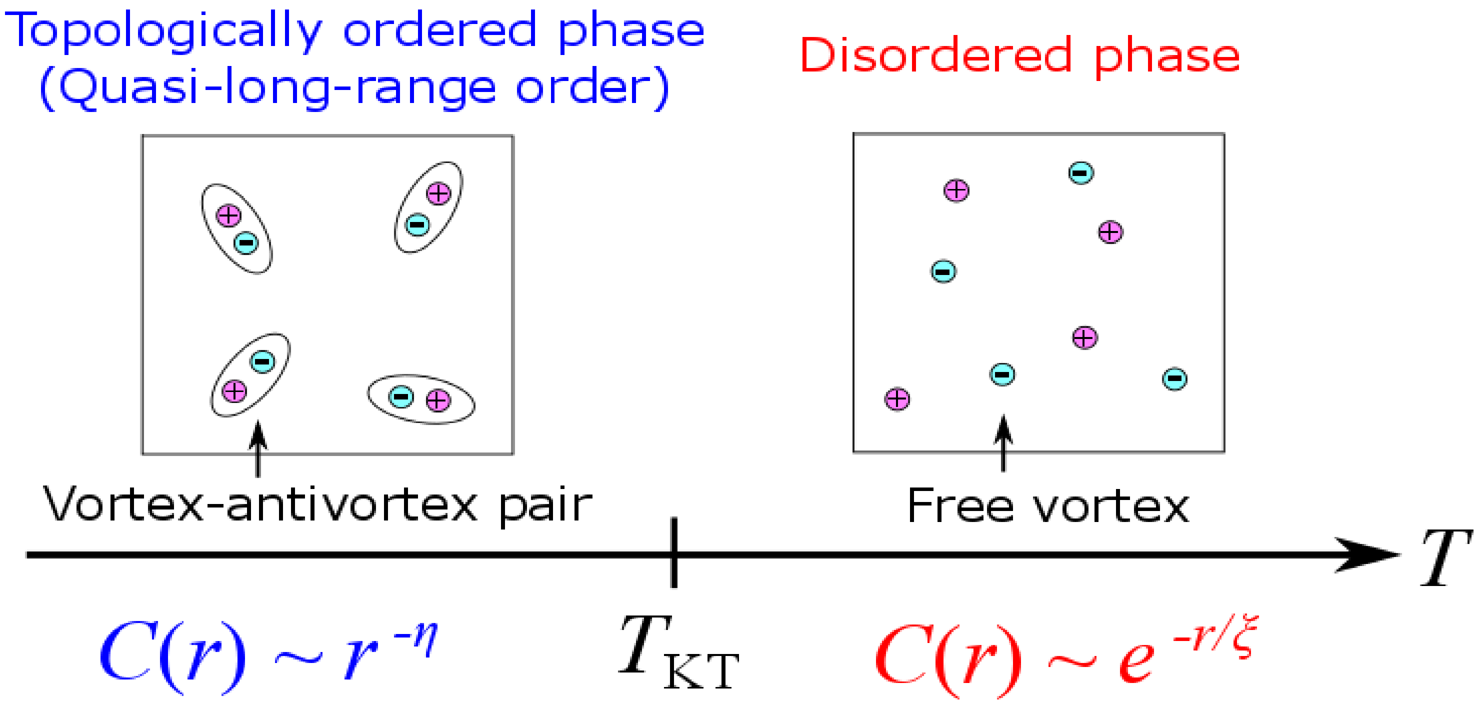
✓ Random force: $F(r;\phi) = -h^1\sin\phi + h^2\cos\phi$

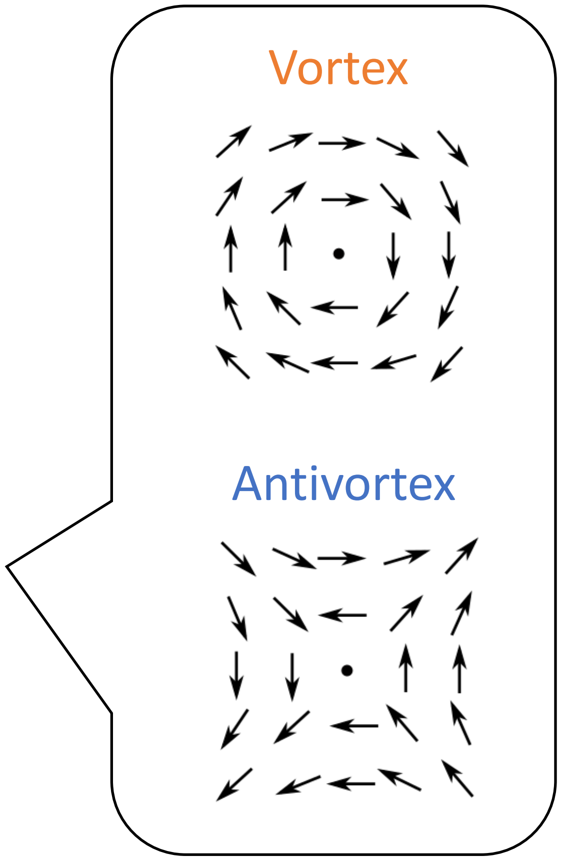
$$\Rightarrow \overline{F(r;\phi)F(r';\phi')} = h_0^2\cos(\phi-\phi')\delta(r-r')$$

Driven random manifold ($N = 1$) with a periodic disorder correlator

Berezinskii-Kosterlitz-Thouless transition

✓ 2D system with U(1) symmetry (XY spin, superfluid film, …)





$\langle \Phi(r) \cdot \Phi(0) \rangle$: spin-spin correlation

BKT transition in 3D DRFXYM

Note: For $N = 1$, the DR is applicable for the driven random manifold.

$\Rightarrow$ 3D driven random manifold = 2D pure manifold with $T_{\mathrm{eff}} = h_0^2/2v$

Spin-wave approximation

3D driven random field XY model = 2D XY model with $T_{\mathrm{eff}} = h_0^2/2v$

For weak disorder, the 3D DRFXYM exhibits quasi-long-range order:

$$G_d(r_1, r_2) := \overline{\Phi(r_1)\Phi(r_2)} - \overline{\Phi(r_1)}\ \overline{\Phi(r_2)} \sim |r_1 - r_2|^{-\eta(\Delta, v)}$$

$\eta(\Delta, v)$: Anomalous exponent depending on disorder Δ and velocity v

For strong disorder, the BKT transition to disordered phase occurs?

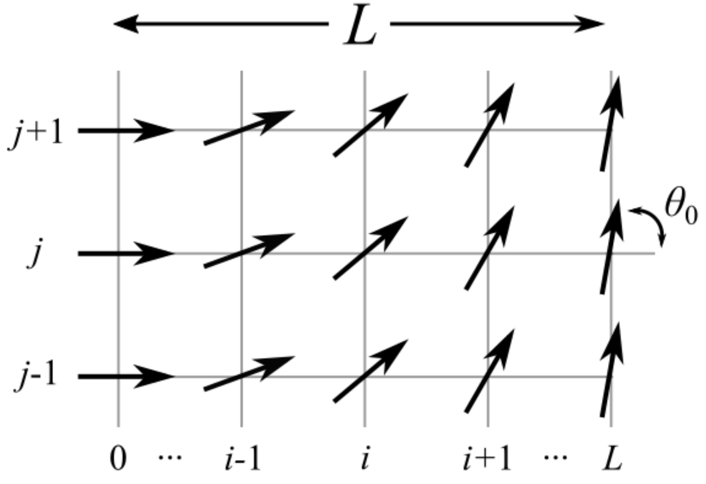
Helicity modulus in 2D XY model

✓ Helicity modulus (2D case)

Twist force at boundary:

$$f(\theta_0) = L\left\langle \sin\left(\theta_{i+1,j} - \theta_{i,j}\right)\right\rangle$$

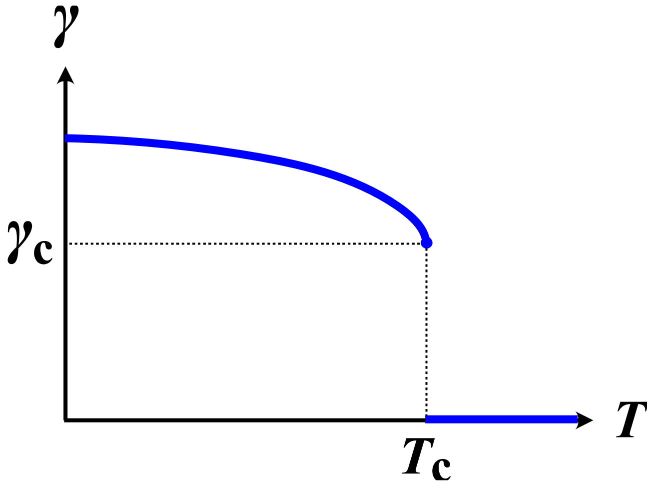
Helicity modulus: $\gamma = \lim_{\theta_0 \to 0} f(\theta_0)/\theta_0$



✓ Discontinuity of γ at BKT transition

$$\lim_{\varepsilon \to 0} \gamma(T_c - \varepsilon) = \gamma_c, \quad \lim_{\varepsilon \to 0} \gamma(T_c + \varepsilon) = 0$$

Universal jump: $\dfrac{T_c}{\gamma_c} = \dfrac{\pi}{2}$



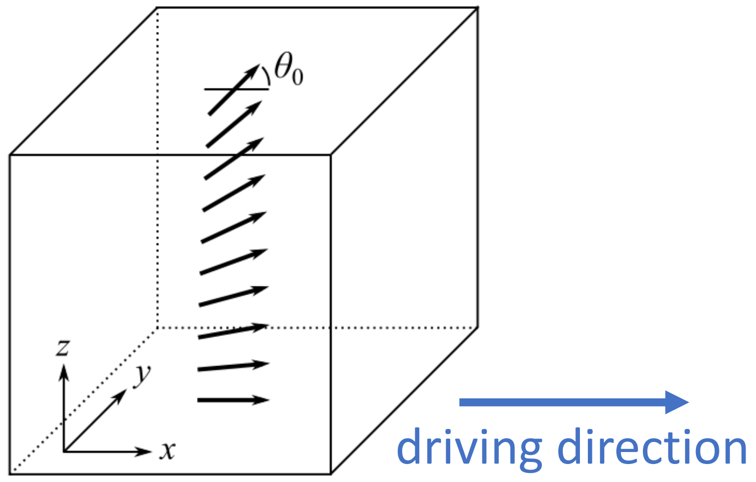
Helicity modulus in 3D-DRFXYM

✓ Helicity modulus (3D case)

Twist force at boundary:

$$f(\theta_0) = L\,\overline{\sin\left(\theta_{i_x,i_y,i_z+1} - \theta_{i_x,i_y,i_z}\right)}$$

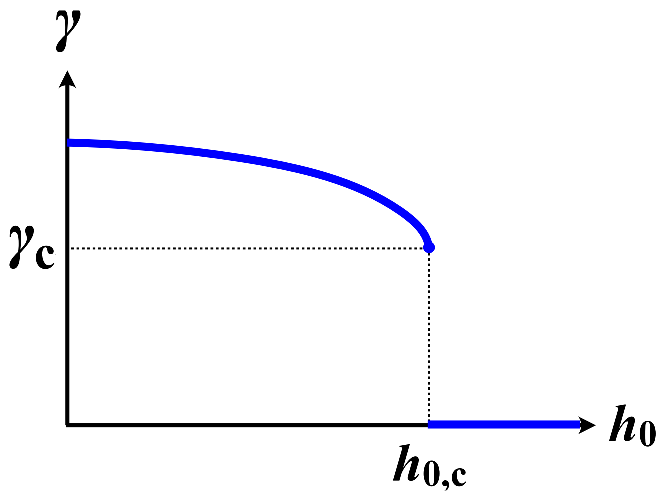
Helicity modulus: $\gamma = \lim_{\theta_0 \to 0} f(\theta_0)/\theta_0$



✓ Discontinuity of γ at 3D BKT transition

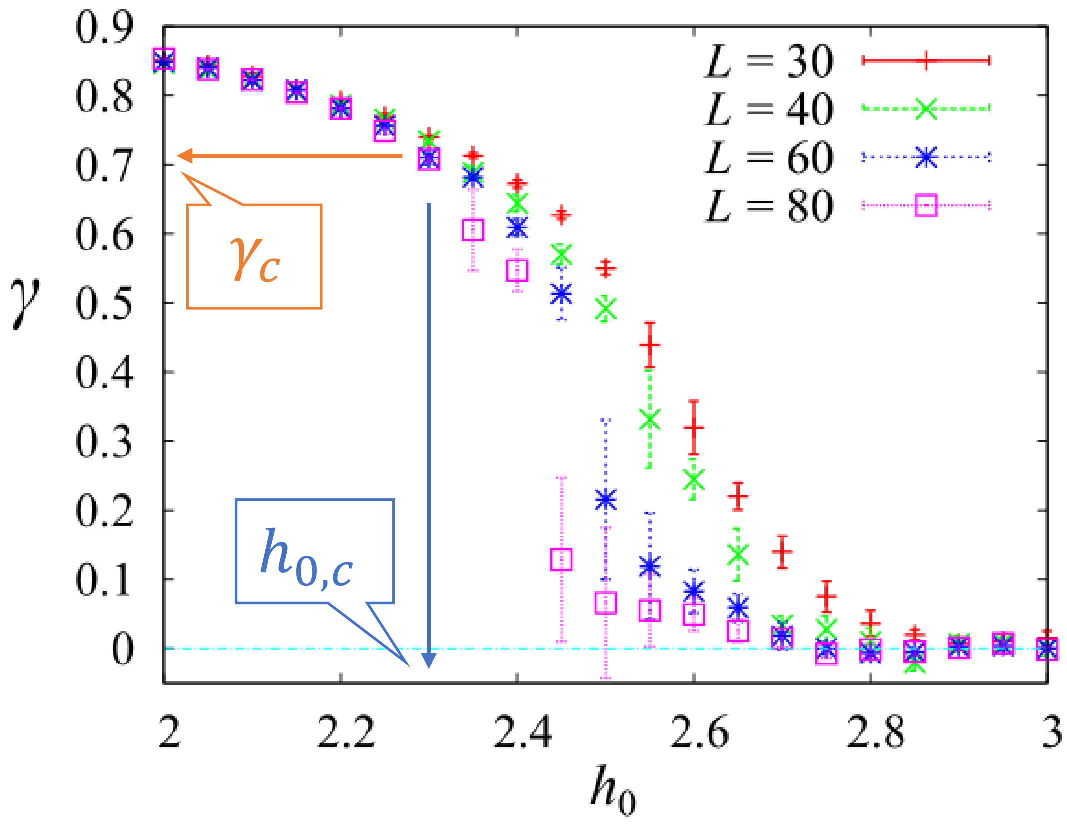
Replacement: $T \to T_{\rm eff} = h_0^2/2v$

Universal jump: $\dfrac{h_{0,c}^2}{\gamma_c v} = \pi$

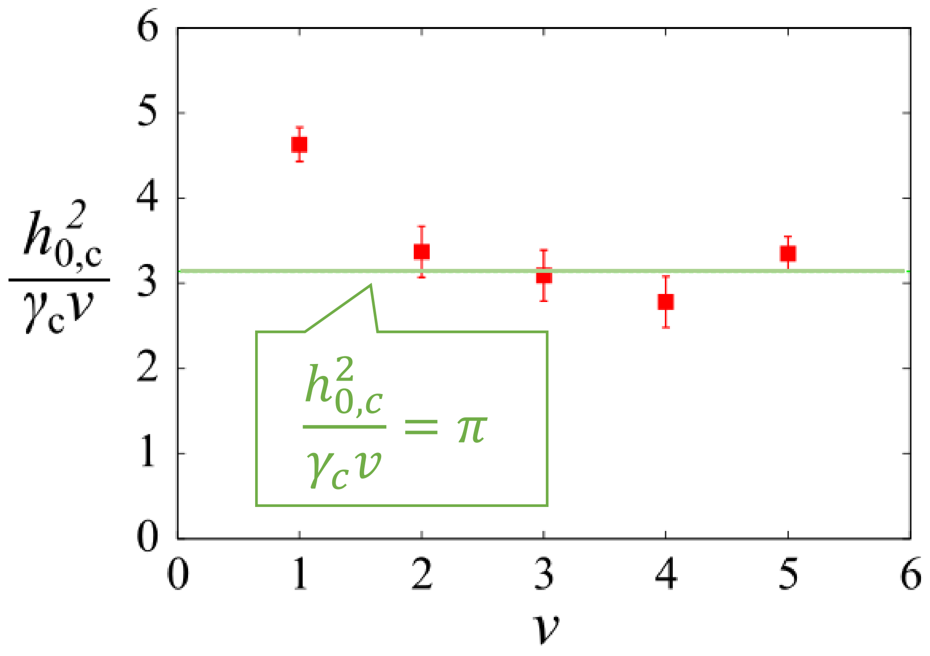


Numerical results (tentative)

✓ Helicity modulus for velocity $v = 2$



✓ Jump of helicity modulus

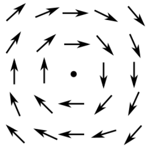


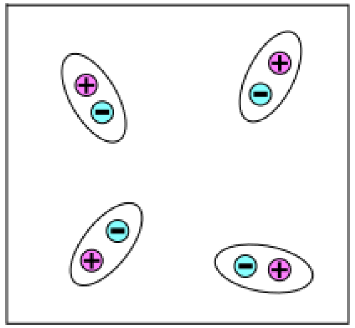
Unpublished

Vortex dissociation picture

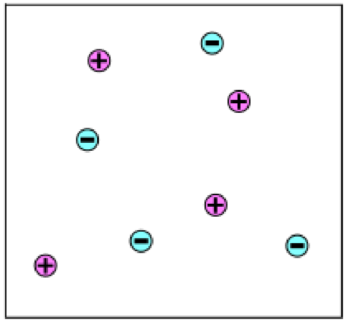
✓ 2D pure XY model

- QLRO phase
  (Low temperature)



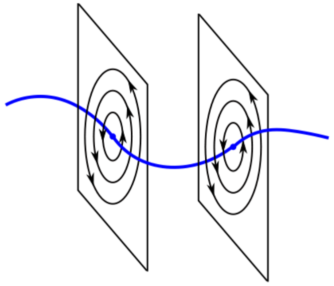


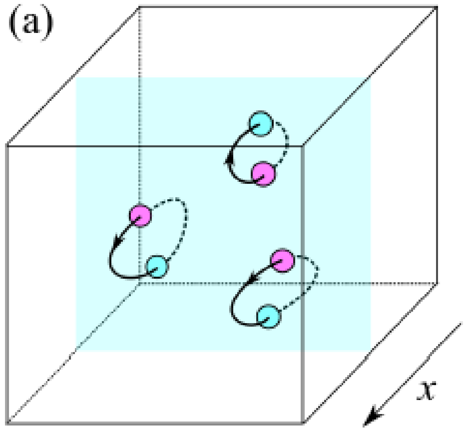
- Disordered phase
  (high temperature)



✓ 3D-DRFXYM

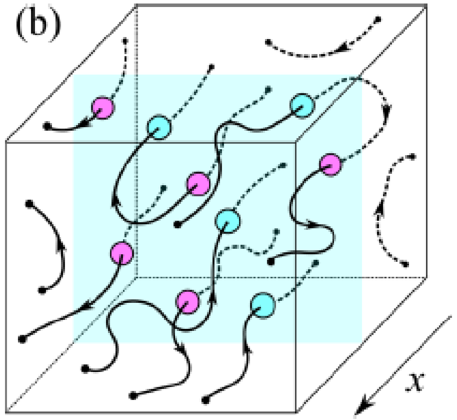
- QLRO phase
  (Weak disorder)





- Disordered phase
  (Strong disorder)

x ⇒ "Time"
Vortex lines ⇒
Trajectory of vortices



Results

Finite-temperature cases

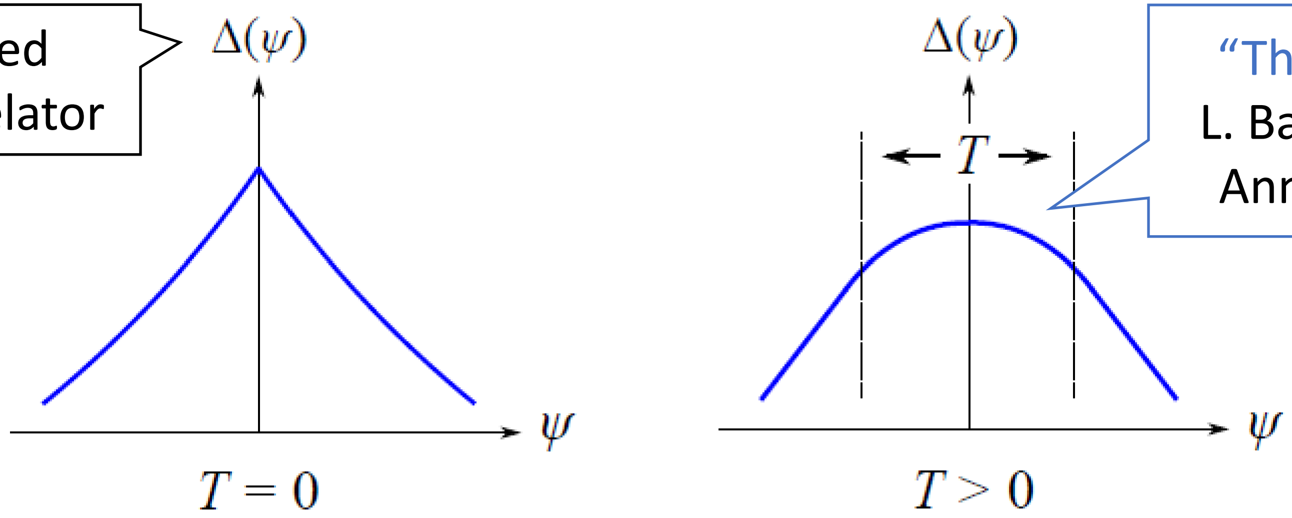
Unpublished

Diffusion effect by thermal fluctuations

Thermal fluctuations ⇒ Diffusion between meta-stable states

⇒ Smoothing out the cusp of disorder correlator $\Delta(\phi)$

Renormalized disorder correlator



"Thermal boundary layer"
L. Balents and P. Le Doussal, Ann. Phys. 315 213 (2005)

✓ RG equation at finite T: $\partial_l \Delta(\phi) = T\partial_l^2 \Delta(\phi) + [\text{zero } T \text{ terms}]$

Diffusion due to thermal fluctuations

Heating effect by nonequilibrium driving

✓ Renormalization of temperature T

- Equilibrium case

$$\partial_l T_l = -(D - 2 + 2\zeta)T_l$$

⇒ No temperature renormalization, except for trivial rescaling

Consequence of the time-reversal symmetry
L. Canet et al., J. Phys. A 44, 495001 (2011)

⇒ Temperature is irrelevant.

⇒ Thermal fluctuations do not affect the large-scale behavior.

- Nonequilibrium case

$$\partial_l T_l = -(D - 2 + 2\zeta_\perp)T_l + O(T_l v^2 \Delta)$$

⇒ Temperature renormalization by nonequilibrium driving

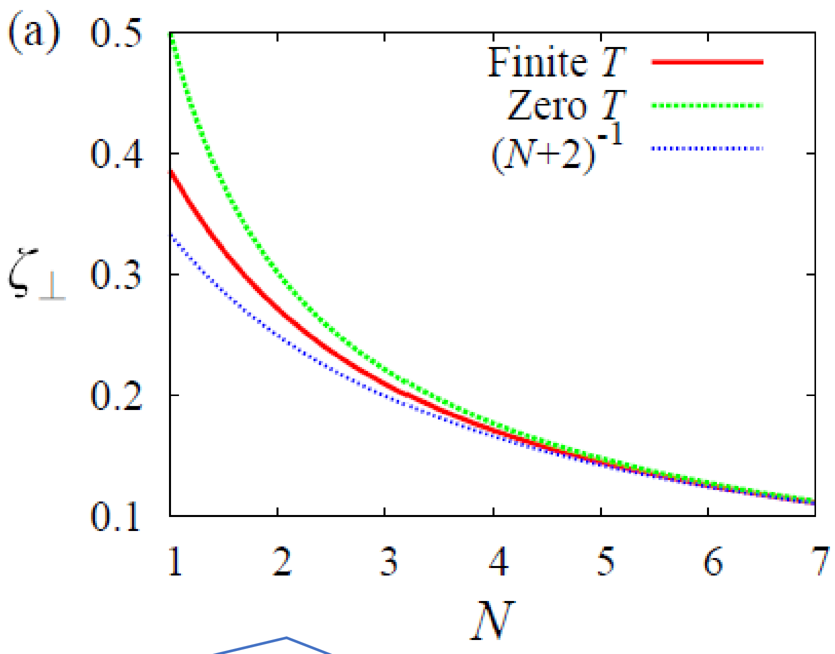
Combination of disorder and driving ⇒ Heating
L. Balents et al., PRB 57, 7705 (1998)

⇒ Temperature becomes relevant.

⇒ Thermal fluctuations can change the large-scale behavior.

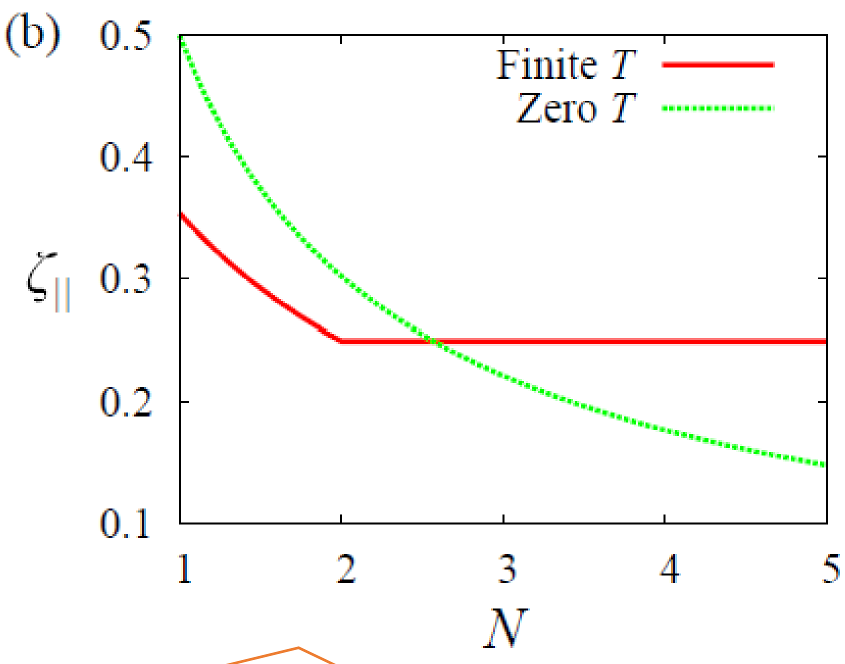
Roughness exponent

✓ Transverse exponent $\zeta_\perp$ at $D = 2$



$\zeta_{\perp,\text{finite}} < \zeta_{\perp,\text{zero}}$ ⇒ The thermal fluctuations reduce the roughness.

✓ Longitudinal exponent $\zeta_\parallel$ at $D = 1$

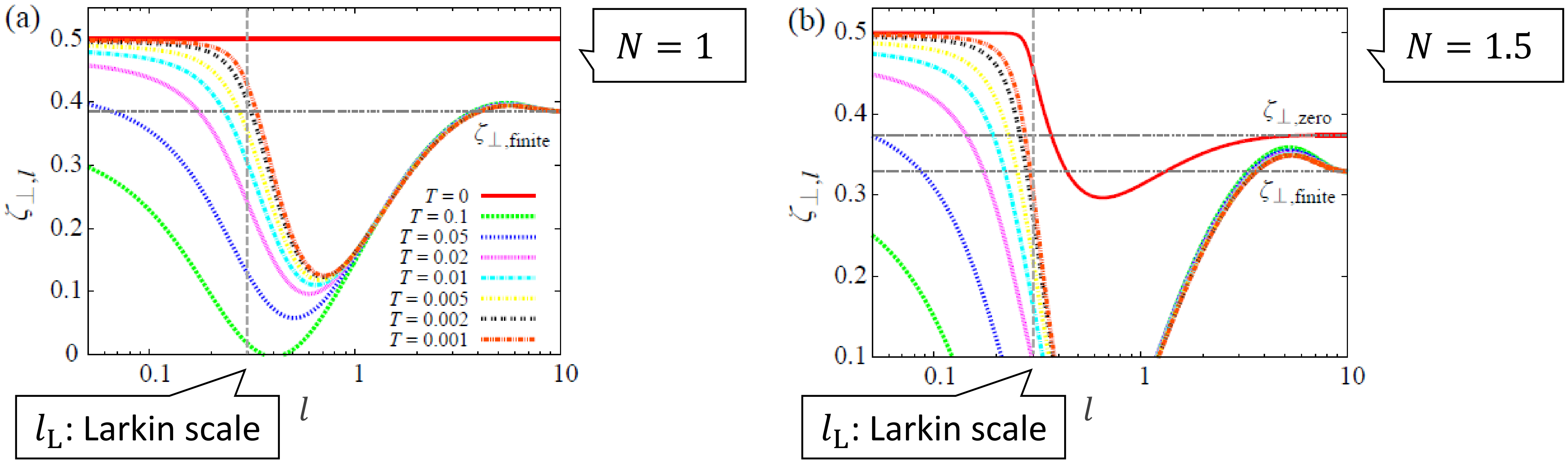


$\zeta_\parallel = 1/4$

$T > 0 \ \rightarrow \ \zeta_\parallel \geq (2 - D)/4$
⇒ $\zeta_{\perp,\text{finite}} > \zeta_{\perp,\text{zero}}$ for large N

Crossover from finite to zero temperature

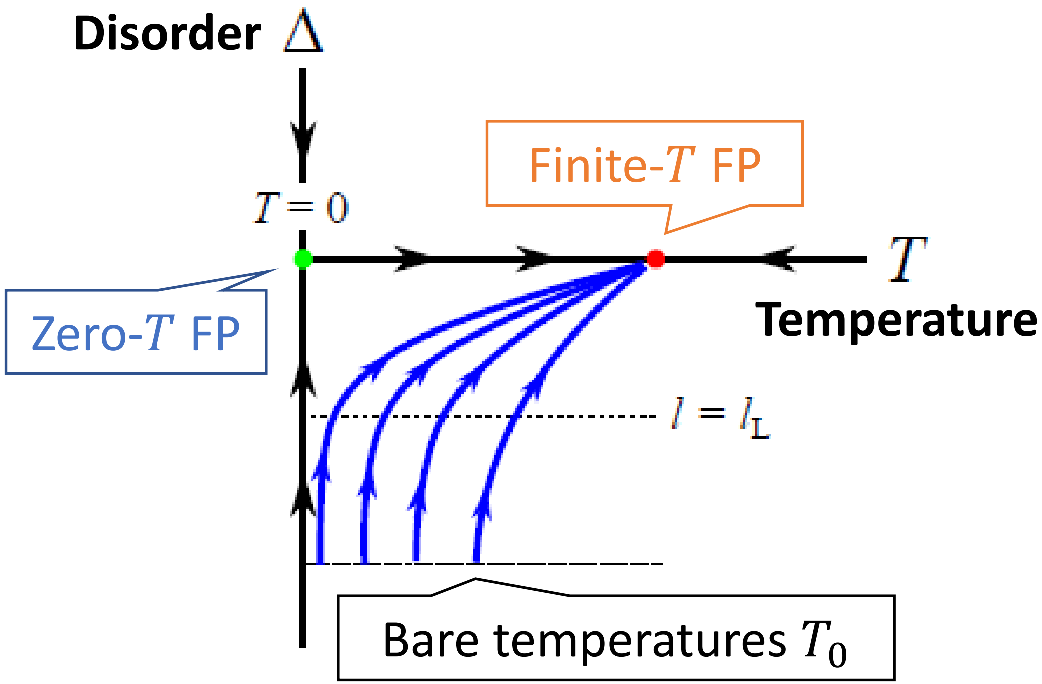
✓ $\zeta_{\perp,l}(T_0)$: Scale-dependent exponent with the bare temperature $T_{l=0} = T_0$



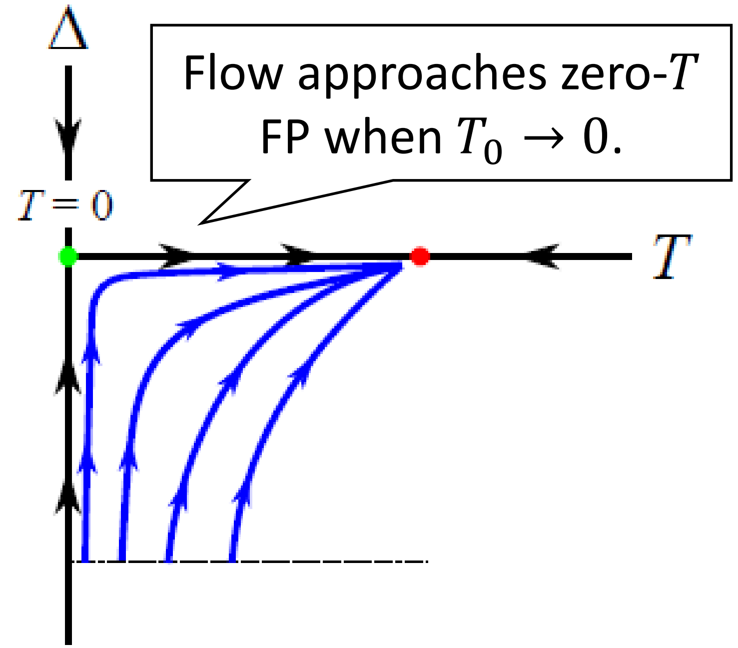
⇒ $\zeta_{\perp,l}(0) \neq \lim_{T_0\to 0} \zeta_{\perp,l}(T_0)$ for $l > l_{\mathrm{L}}$ (l_{L}: scale at which $\Delta(\phi)$ develops a cusp)

Crossover of RG flow in disorder-temperature space

✓ Actual RG flow



✓ Naïve expectation

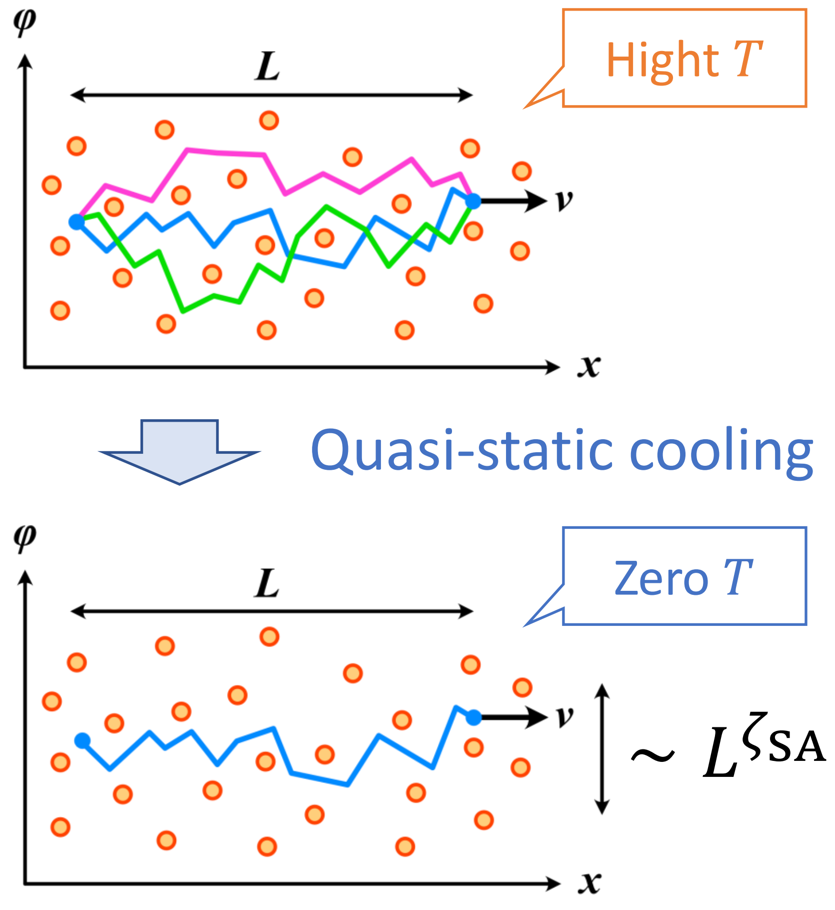


Rapid increase of T_l at $l = l_{\mathrm{L}}$ (“temperature explosion”)

⇒ Zero-T FP is not accessible in the limit $T_0 \to 0$!

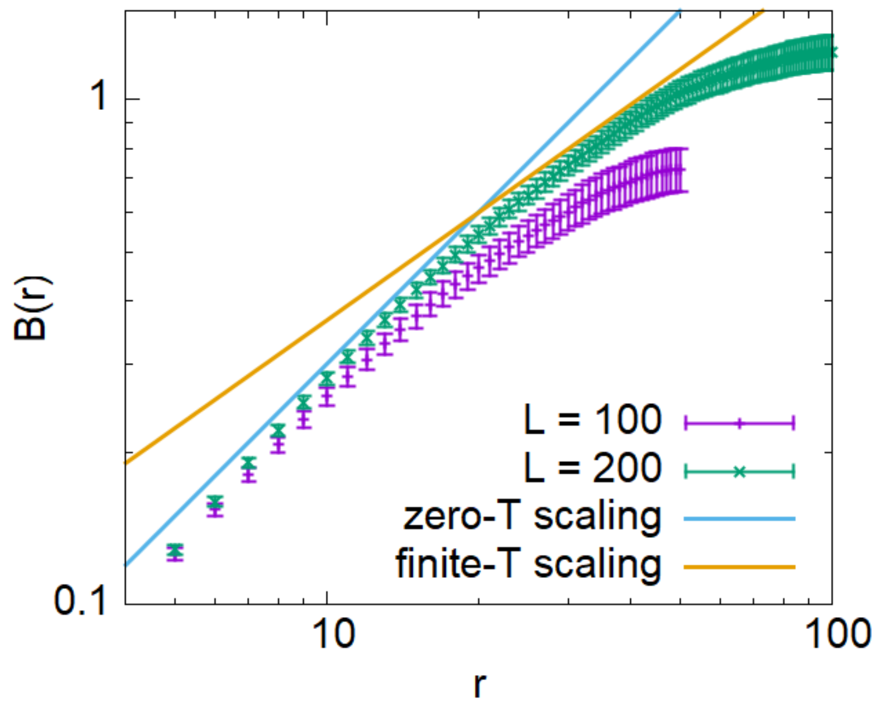
Numerical simulation for $N = D = 1$ (tentative)

✓ Simulated annealing



✓ Question: $\zeta_{\mathrm{SA}} = \zeta_{\perp,\mathrm{zero}}$ or $\zeta_{\perp,\mathrm{finite}}$?

MSD: $B(r) = \overline{\langle[\phi(r) - \phi(0)]^2\rangle}$



Summary

✓ Zero temperature case

- Dimensional reduction:

D-dim. driven disordered system at $T = 0$ $\cong$ $(D-1)$-dim. pure system at $T \neq 0$

- To what extent does the DR fail? $\Rightarrow$ Non-analyticity of renormalized disorder
- BKT transition in 3D driven random field XY model

✓ Finite temperature case

- Heating effect by nonequilibrium driving $\Rightarrow$ Importance of temperature
- Anomalous crossover from finite- to zero-temperature RG flow
  $\Rightarrow$ The zero-T scaling is not observable in the real world?

Appendix

Dynamical formulation of FRG

✓ Langevin dynamics:

$$\partial_t \phi = f(\phi) + \xi, \qquad \langle \xi(r,t)\xi(r',t')\rangle = 2T\delta(r-r')\delta(t-t')$$

$$\Rightarrow Z[J,\hat{J}] = \int \mathcal{D}\phi\mathcal{D}\hat{\phi}\, \exp\left(-S[\phi,i\hat{\phi}] + J\cdot\phi + \hat{J}\cdot i\hat{\phi}\right)$$

$$S[\phi,i\hat{\phi}] = \int drdt \left[i\hat{\phi}\{\partial_t\phi - f(\phi)\} - T\left(i\hat{\phi}\right)^2\right]$$

See [L. Canet et al., J. Phys. A 44, 495001 (2011)] for details

✓ Scale-dependent mass term:

$$\Delta S_k[\phi,i\hat{\phi}] = \frac{1}{2}\int dqd\omega\, R_k(q) i\hat{\phi}(q,\omega)\phi(-q,-\omega)$$

$$\psi = \frac{\delta \ln Z_k[J,\hat{J}]}{\delta J}$$

$$i\hat{\psi} = \frac{\delta \ln Z_k[J,\hat{J}]}{\delta \hat{J}}$$

⇒ Scale-dependent effective action: $\Gamma_k\left[\psi, i\hat{\psi}\right]$