Gradient Flow Exact Renormalization Group and Scalar Field Theories

Junichi Haruna
(Kyoto University)

2023/1/21 @ FRG at RIKEN 2023

Abstract

- We discuss some aspects of Gradient Flow Exact Renormalization Group, a new frame work to define the Wilsonian effective action via the gradient flow
- We study the fixed point structure of GFERG flow associated with a general polynomial gradient flow equation for scalar field theories
- We also investigate the RG flow of $O(3)$ non-linear sigma model with the Wess-Zumino term by GFERG, as a loophole of the above discussion

Contents

Contents

Review of
Gradient Flow Exact
Renormalization Group

Exact Renormalization Group (ERG)

- A framework to study physics under varying the energy scale

- The Wilson action S_τ is intuitively defined by integrating out higher momentum modes of the fields:

$$e^{-S_\tau} := \int D\phi_{p>\Lambda} e^{-S_0}$$

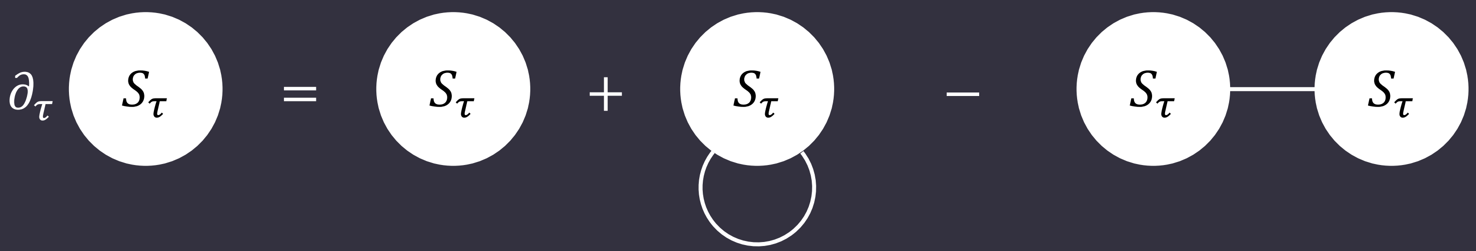
($\Lambda := \Lambda_0 e^{-\tau}$, Λ_0: cutoff)

- τ-dependence of the Wilson action S_τ is described by a differential equation → "ERG equation"

Wilson-Polchinski equation

- A typical example of the ERG equation:

$$\partial_\tau S_\tau = \int_p \left\{ \left[\left(2p^2 + \frac{D+2-\eta_\tau}{2} \right) + p_\mu \frac{\partial}{\partial p_\mu} \right] \phi_i(p) \frac{\delta S_\tau}{\delta \phi_i(p)} + \left(2p^2 + 1 - \frac{\eta_\tau}{2} \right) \left(\frac{\delta^2 S_\tau}{\delta \phi_i(p) \delta \phi_i(-p)} - \frac{\delta S_\tau}{\delta \phi_i(p)} \frac{\delta S_\tau}{\delta \phi_i(-p)} \right) \right\}$$



- This equation defines a renormalization procedure non-perturbatively

(η_τ: anomalous dimension) [J.Polchinski *Nucl.Phys.B* 231 (1984) 269-295]

(We work on the dimensionless framework and D-dimensional Euclidean space)

Gauge invariance in ERG

- The Wilson action with the naive UV cutoff is inconsistent with gauge invariance

- The gauge transformation mixes higher and lower momentum modes:

$$A_\mu^a(p) \to A_\mu^a(p) - p_\mu \omega^a(p) - i f^{abc} \int_q \omega^b(p-q) A^c(q)$$

- Can we define a Wilson action in a manifestly gauge invariant manner?

Renormalization and Diffusion

- The solution to the WP equation can be written in the following form:

$$e^{S_\tau[\phi]} = \hat{s}_\phi^{-1} \int D\phi' \prod_{x,i} \delta(\phi_i(x) - e^{\tau(D-2)/2} Z^{1/2} \varphi_i'(t, xe^\tau))\, \hat{s}_{\phi'} e^{S_{\tau=0}[\phi']}$$

($\hat{s}_\phi \coloneqq \exp\left(-\frac{1}{2}\int_x \frac{\delta^2}{\delta\phi_i(x)^2}\right)$: "scrambler")

- φ' is a solution to the diffusion equation:

$$\partial_t \varphi'(t,x) = \partial_x^2 \varphi'(t,x),\ \ \varphi'(0,x) = \phi'(x)$$

where $t \coloneqq e^{2\tau} - 1$

- This representation implies that the coarse-graining by the diffusion can be used to define an RG flow

Gradient Flow (GF)

- This one-parameter deformation of fields via the diffusion equation has been studied in the context of "gradient flow"

- The gradient flow is a method to construct composite operators without the equal-point singularity

- Correlation functions are UV finite with wave function renormalization:

$$Z_t^{-n/2} \langle \varphi(t,x_1)\varphi(t,x_2)\cdots\varphi(t,x_n)\rangle_\phi < \infty$$

  even for the equal point case (e.g. $x_1 = x_2$)

[F.Capponi,L.Debbio,S.Ehret,R.Pellegrini,A.Rago 1512.02851]

Gradient Flow ERG (GFERG)

- GF equation for gauge fields [M.Lüscher, P.Weisz 1101.0963]

$$\partial_t B'_\mu = D'_\nu G'_{\nu\mu} + \alpha_0 D'_\mu \partial_\nu B'_\nu$$

with $B'_\mu(0,x) = A'_\mu(x)$

- S_τ for gauge fields can be defined via the GF equation [H.Sonoda and H.Suzuki 2012.03568]

$$e^{-S_\tau[A^a_\mu]} \coloneqq \hat{s}_A^{-1} \int [DA'^a_\mu] \prod_{x',a,\mu} \delta\left(A^a_\mu(x) - e^{\tau(D-2)/2} B'^a_\mu(t, x' e^\tau)\right) \hat{s}_{A'} e^{-S_{\tau=0}[A'^a_\mu]}$$

($\hat{s}_A \coloneqq \exp\left[-\frac{1}{2}\int_x \frac{\delta^2}{\delta A^a_\mu(x)\delta A^a_\mu(x)}\right]$: "scrambler" for gauge fields)

- GFERG defines an RG flow in a gauge-invariance way!

Contents

Fixed Point Structure of GFERG for Scalar Field Theories

Based on arXiv:2201.04111, PTEP 2022, No.3, 033B03 (2022) with Y.Abe (Wisconsin), Y.Hamada (KEK)

Our motivation

- The appropriate gradient flow eq. highly depends on details of the theory, such as its symmetry and interactions
- The GFERG flow depends on the form of the GF eq., and then becomes different for each theory
- This means that the GFERG eq. for general scalar field theories is no longer given by the WP eq.
- Is GFERG consistent with the conventional ERG?

Our motivation

- The gradient flow equation for the $O(N)$ non-linear sigma model

$$\partial_t \varphi_i = \partial_x^2 \varphi_i - (\varphi_j \partial_x^2 \varphi_j)\varphi_i$$

[H.Makino, H.Suzuki 1410.7538]

- But this model and the linear sigma model in three dimension belong to the same universality class

- "Does GFERG give the same prediction as the conventional ERG for the IR behavior of a theory?"

- These facts strongly motivate us to study the fixed points and the critical exponents in GFERG

Our work

- We study the fixed points of the general GFERG eq. for scalar fields
  → The fixed points of the WP eq. appear in the $\tau \to \infty$ limit along the GFERG flow

- We show the GFERG eq. has a similar RG flow structure around a fixed point
  → Scaling dimensions of relevant or marginal operators are the same, while those of irrelevant operators can be different

- GFERG gives the same prediction as the conventional ERG for the IR behavior of a theory

General gradient flow equation

- General form of gradient flow equations

$$\partial_\tau \varphi_i = \underbrace{\partial_x^2 \varphi_i}_{\text{WP part}} + \sum_{n=n_{\min}}^{\infty} \int_{x_1,\dots,x_n} \underbrace{f_i^{i_1,\dots,i_n}(x; x_1, \dots, x_n; \partial_{x_1}, \dots, \partial_{x_n}) \varphi_{i_1}(\tau, x_1) \cdots \varphi_{i_n}(\tau, x_n)}_{\text{extra terms}}$$

- Counterpart of WP eq. in GFERG (GFERG eq.)

$$\partial_\tau e^{-S_\tau[\Phi]} = (\text{WP part}) + \sum_{n=n_{\min}}^{\infty} \lambda^{n-1}(\tau) \int_{x,x_1,\dots,x_n} \frac{\delta}{\delta \phi_i(x)} \left\{ f_i^{i_1,\dots,i_n} \left(\phi_{i_1}(x_1) + \frac{\delta}{\delta \phi_{i_1}(x_1)} \right) \times \cdots \times \left(\phi_{i_n}(x_n) + \frac{\delta}{\delta \phi_{i_n}(x_n)} \right) \right\} e^{-S_\tau[\Phi]}$$

where $\boxed{\lambda(\tau) := e^{-\tau(D-2)/2} Z_\tau^{-1/2}}$

Fixed point

- The fixed-point action S^* is defined as

$$\partial_\tau S^* = 0 \text{ and } S^* \coloneqq \lim_{\tau\to\infty} S_\tau$$

- S^* satisfies

$$0 = (\text{WP part}) + \sum_{n=n_{\min}}^{\infty} \lambda^{n-1}(\infty) \int_{x,x_1,\ldots,x_n} \frac{\delta}{\delta\phi_i(x)} \left\{ f_i^{i_1,\ldots,i_n} \left(\phi_{i_1}(x_1) + \frac{\delta}{\delta\phi_{i_1}(x_1)} \right) \times \cdots \times \left(\phi_{i_n}(x_n) + \frac{\delta}{\delta\phi_{i_n}(x_n)} \right) \right\} e^{-S^*[\Phi]}$$

- Asymptotic behavior of $\lambda(\tau)$

$$\lambda(\tau) \sim \exp(-\tau(D-2+\eta)/2) \qquad \frac{\eta}{2} \coloneqq \frac{d}{d\tau} \log Z_\tau \Big|_{\tau=\infty}$$

- $\lambda(\infty)$ should vanish from the cluster decomposition principle at S^*
$\Rightarrow S^*$ satisfies the fixed-point condition of the WP eq.

RG flow around fixed point

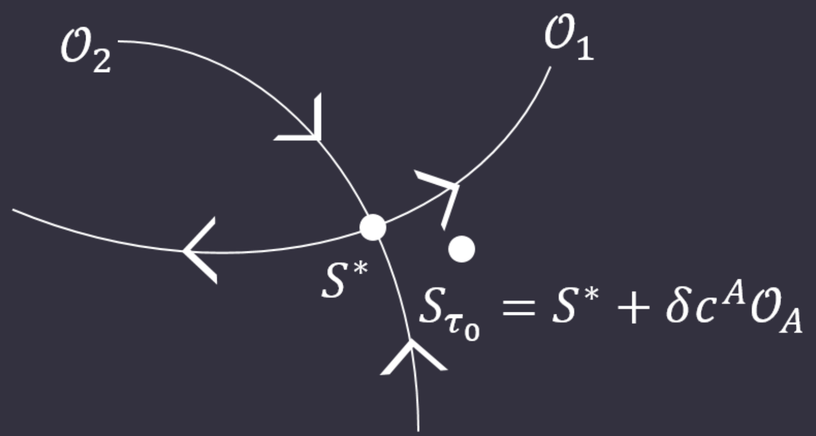
- Let us study the τ-dependence of S_τ after a long time $\tau = \tau_0 \gg 1$ so that $\lambda(\tau_0) \sim e^{-\tau_0(D-2+\eta)/2} \ll 1$

- Consider perturbating S_τ from a fixed point S^* at $\tau = \tau_0$ as

$$S_{\tau_0} = S^* + \sum_A \delta c^A \mathcal{O}_A$$

$|\delta c^A| \ll 1$

$\mathcal{O}_A$: a complete set of operators (defined in the next slide)



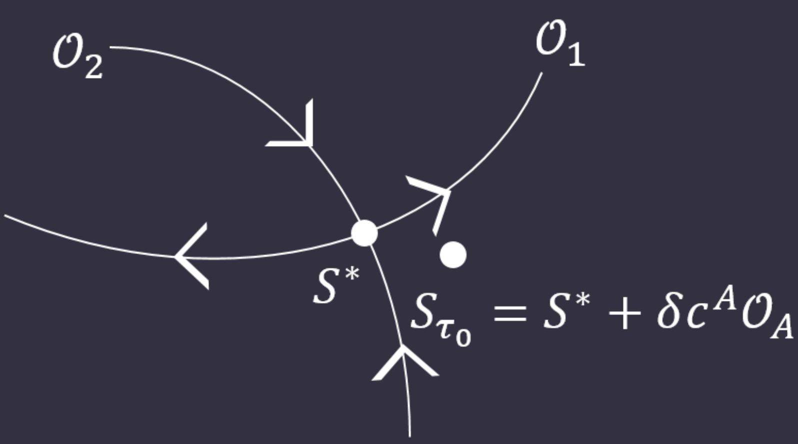
Solution in the $\tau \to \infty$ limit

- We have two small quantities: δc^A, $\lambda_0 := e^{-\tau_0(D-2+\eta)/2}$

- Solution to the GFERG eq. up to the leading order

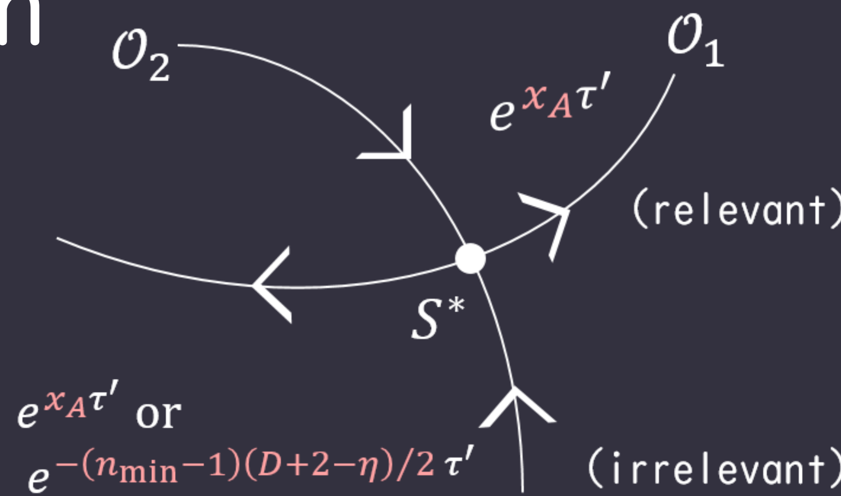
$$S_\tau = S^* + \sum_A \left(\delta c^A e^{x_A \tau'} - \lambda_0^{n_{\min}-1}\left(e^{-(n_{\min}-1)(D-2+\eta)\tau'/2} - e^{x_A \tau'}\right) h^A\right) \mathcal{O}_A$$

where $\tau = \tau_0 + \tau'$



$\mathcal{O}_A$: eigenoperators of the linearized WP eq. around S^*
x_A: eigenvalue of $\mathcal{O}_A$
$n_{\min}$: minimum order of the non-linear terms in the GF eq.
h^A: some constant

Scaling dimension



- Scaling Dimension d_A

$$d_A = \max(x_A, -(n_m - 1)(D - 2 + \eta)/2)$$

$x_A \geq 0$ (relevant or marignal) $\implies$ $d_A = x_A$

$x_A < 0$ (irrelevant) $\implies$ $d_A = -\min(|x_A|, (n_m - 1)(D - 2 + \eta)/2)$

- d_A of relevant or marginal operators are x_A
  $\Rightarrow$ critical exponents are the same as those of the WP eq.

Contents

GFERG analysis of O(3) non-linear sigma model in two dimensions

Based on a preliminary work with Hiroshi Suzuki (Kyushu)

Our Motivation

- $O(3)$ non-linear sigma model is a loophole of the previous study

- This model is gapless with the Wess-Zumino term with $\theta = \pi$ → non-trivial fixed point

- Its gradient flow equation

$$\partial_t \varphi_i = \partial_x^2 \varphi_i - \left(\varphi_j \partial_x^2 \varphi_j\right) \varphi_i$$

[H.Makino, H.Suzuki 1410.7538]

- It is interesting to study its RG flow by GFERG

Wilson action

- We want to treat bare fields rather than renormalized ones
- We should rescale $\phi_i(x) \to \lambda(\tau)^{-\frac{1}{2}}\phi_i(x)$ in the definition of the Wilsonian effective action of GFERG:

$$e^{S_\tau[\phi]} = \hat{s}^{-1}_{\lambda(\tau)^{-1/2}\phi} \int D\phi' \prod_x \delta(\phi_i(x) - \varphi'_i(t, xe^\tau))\, \hat{s}_{\phi'} e^{S_{\tau=0}[\phi']}$$

where φ'_i is the solution to the GF equation:

$$\partial_t \varphi'_i = \partial_x^2 \varphi'_i - \left(\varphi'_j \partial_x^2 \varphi'_j\right)\varphi'_i$$

Constraint along RG flow

- Constraint of the fields of $O(N)$ NL sigma model

$$\phi_i^2 = 1$$

- S_τ preserves it in the sense of the (modified) correlation functions:

$$\ll (\phi_i^2 - 1)\phi_{i_1} \cdots \phi_{i_n} \gg_{S_\tau} = 0$$

where $\ll \mathcal{O}[\phi] \gg_{S_\tau} := \int D\phi_i \left(\hat{s}_{\lambda(\tau)^{-1/2}\phi}\mathcal{O}[\phi]\right) e^{S_\tau[\phi]}$

(Recall: $e^{S_\tau[\phi]} = \hat{s}^{-1}_{\lambda(\tau)^{-1/2}\phi} \int D\phi' \prod_x \delta(\phi_i(x) - \varphi_i'(t, xe^\tau))\, \hat{s}_{\phi'} e^{S_{\tau=0}[\phi']}$)

- GFERG can define an RG flow which preserves the information of the target space

GFREG equation

- GFERG equation

$$\partial_\tau e^{S_\tau} = (\text{WP part}) +$$

$$2\int_x \frac{\delta}{\delta\phi_i(x)}\left(\phi_j(x) + \lambda(\tau)^2 \frac{\delta}{\delta\phi_j(x)}\right)\partial^2\left(\phi_j(x) + \lambda(\tau)^2 \frac{\delta}{\delta\phi_j(x)}\right)\left(\phi_i(x) + \lambda(\tau)^2 \frac{\delta}{\delta\phi_i(x)}\right)e^{S_\tau}$$

- We assume $\lambda(\infty)$ takes a finite non-zero value g
- Fixed point condition

$$0 = (\text{WP part}) +$$

$$+2\int_x \frac{\delta}{\delta\phi_i(x)}\left(\phi_j(x) + g^2 \frac{\delta}{\delta\phi_j(x)}\right)\partial^2\left(\phi_j(x) + g^2 \frac{\delta}{\delta\phi_j(x)}\right)\left(\phi_i(x) + g^2 \frac{\delta}{\delta\phi_i(x)}\right)e^{S^*}$$

Ansatz

$$S^* = \frac{1}{2g^2}\int_p \frac{p^2+m^2}{e^{-2p^2}+p^2+m^2}\phi_i(-p)\,\phi_i(p)$$

$$+\frac{i\theta}{3}\int_x \epsilon^{ijk}\epsilon_{\mu\nu}\widetilde{\Phi}_i(x)\partial_\mu\widetilde{\Phi}_j(x)\partial_\nu\widetilde{\Phi}_k(x)$$

$$+\frac{h}{4}\int_x \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\partial_\mu\widetilde{\Phi}_i(x)\partial_\nu\widetilde{\Phi}_j(x)\,\partial_\rho\widetilde{\Phi}_i(x)\partial_\sigma\widetilde{\Phi}_j(x)$$

where $\widetilde{\Phi}_i(p) \coloneqq \frac{e^{-p^2}}{e^{-2p^2}+p^2+m^2}\phi_i(p)$

- We consider only mass and kinetic terms, and these cubic and quartic interactions ("truncation method")

Fixed Point Condition

$$
\begin{cases}
0 = -g^{-2}m^2 + 2(3m^2 - 1)c_1 \\
0 = 2g^2 h c_2 + 2(m^2 - 4)c_0 + 12c_1 - 10g^4 h c_3 \\
0 = \theta(m^2 + g^2(6 - 4m^2)c_1) \\
0 = \frac{h}{2}(1 + m^2)\big(1 + 5m^2 + 8g^2\,(2(m^2 - 1)c_0 + c_1)\big) + 2\theta^2(g^2 - 10g^4\theta^2 c_1)
\end{cases}
$$

where $c_i \ (i = 1,2,3)$ is some constant including m^2

- These constants arise from the loop integrals such as

$$c_0 \coloneqq \int_p \frac{e^{-2p^2}}{e^{-2p^2} + p^2 + m^2}$$

Fixed Point

- Unfortunately, the solution to the all fixed point condition is just the Gaussian one:

$$g = \theta = h = 0$$

- If we neglect the condition from the mass term, there is a non-trivial fixed point:

$$m^2 \simeq 2.82, g^{-2} \simeq 9.29 \times 10^{-3}, h \simeq -2.39 \times 10^3$$

with $\theta = 3/4$

- We can discuss $\theta = \pi$ can be a fixed point from the periodicity $(\theta \to \theta + 2\pi)$ and CP symmetry → Our ansatz may be bad..

Contents

Conclusions and Future Directions

Conclusion I

- Studied the fixed point structure of the GFERG equation associated with a general gradient flow equation for scalar field theories
- Showed that the fixed points are the same as those of the Wilson-Polchinski equation in general
- Discussed that the GFERG equation has a similar RG flow structure around a fixed point to the WP equation
- GFERG gives the same prediction as the conventional ERG for the IR behavior of a theory

Conclusion 2

- Studied an RG flow of $O(3)$ non-linear sigma model in two-dimensions
- Wrote down its GFERG equation, which preserves the constraint $\phi_i^2 = 1$, i.e., the information of the target space
- Discussed that the fixed point and found there is a non-trivial fixed point if we neglect the condition for the mass term, as well as the Gaussian one

Future Direction

- (Non-abelian) gauge theory
- Gravity and asymptotic safety
  "Wilson action with manifest diffeomorphism invariance"
- Scalar field theories with a non-trivial target space in two dimensions
  e.g.) $O(N)$ NL sigma model, $\boldsymbol{CP}^{N-1}$ model
- Effects of topological terms in QFT

Thank you!