Nonperturbative Anomaly and Functional RG

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Functional Renormalization Group

$\Gamma_k[\phi]$: scale (k) - dependent effective action

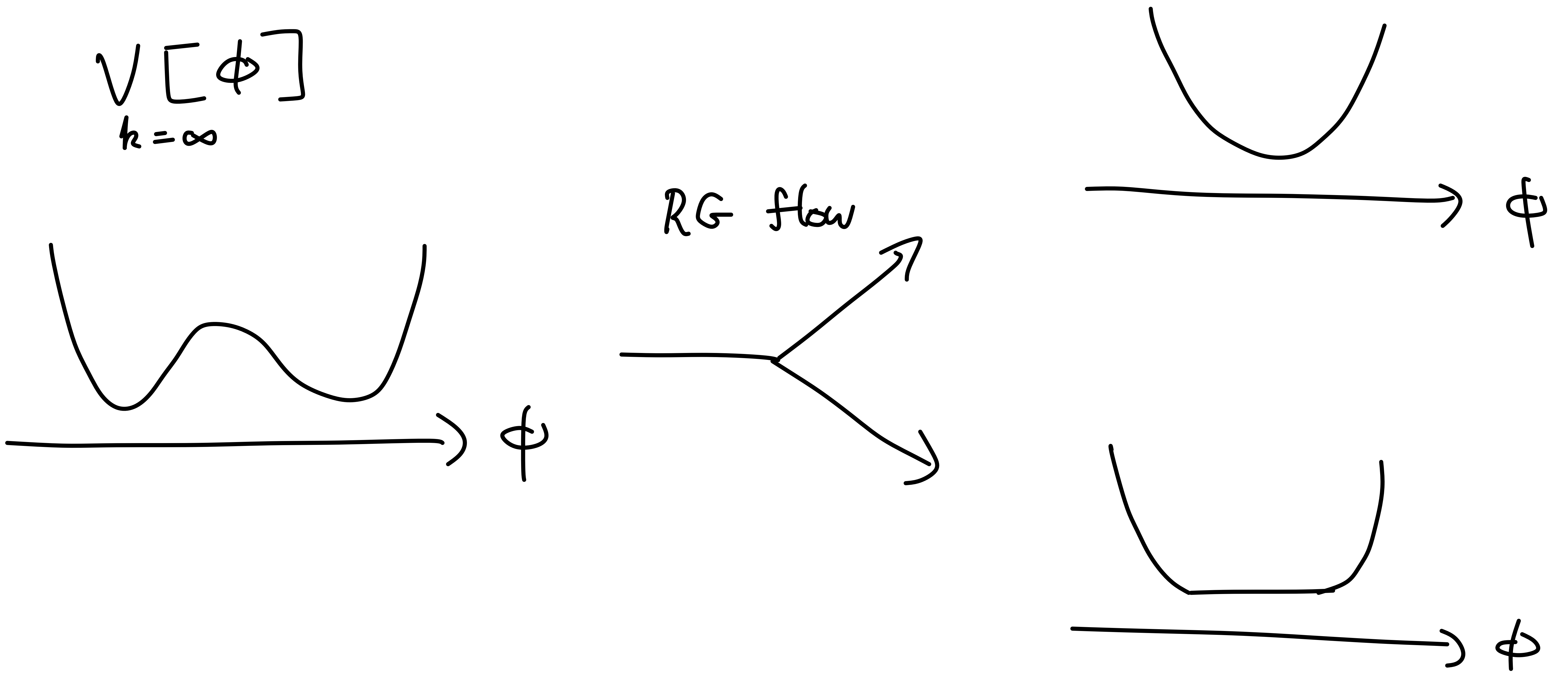
$$\begin{cases} k\partial_k \Gamma_k = F\left[\Gamma_k, \frac{\delta}{\delta\phi}\Gamma_k, \frac{\delta^2}{\delta\phi\delta\phi}\Gamma_k, \cdots\right] & \text{(flow eq.)} \\ \Gamma_{k=\infty} = S_{\text{classical}} & \text{(initial condition)} \end{cases}$$

$\Longrightarrow$ $\Gamma_{k=0}[\phi]$: quantum effective action.

Combined with some truncations (based on good physical intuitions)

FRG gives a useful computational framework to study field theories.

Conventional Phases of Matters



Symmetric phase

$\langle \phi \rangle = 0$

Broken phase

$\langle \phi \rangle = \pm v$

In many cases, FRG can discuss SSB by computing local effective potentials of order parameters.

Topological Phases of Matters

In quantum field theories (QFTs), conventional classification by local order parameters is not enough to characterize phases.

(typical examples)

(Intrinsic) Topological Order

- Topological degeneracy of ground states
- Deconfined gauge fields in the IR limit

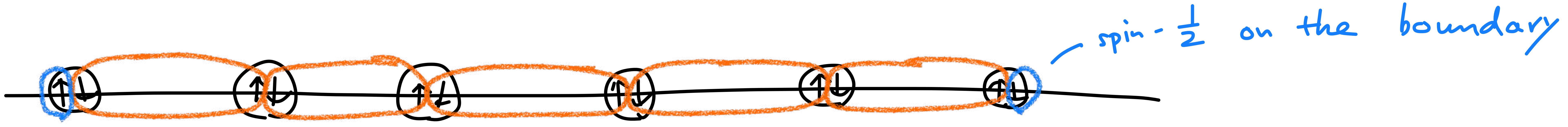
Symmetry-Protected Topological States

- Trivially gapped state on closed space
- Nontrivial degeneracy on the boundaries

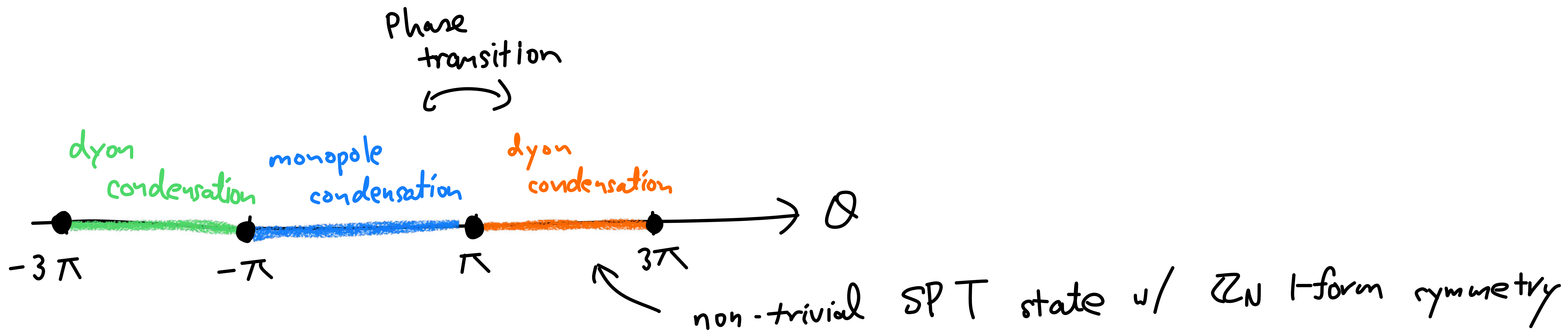
How can FRG treat these systems?

Examples of SPTs

- spin-1 Heisenberg chain $\quad H = J \sum_n \vec{S}_n \cdot \vec{S}_{n+1} \quad (\simeq 2d\ \mathbb{CP}^1\ \sigma\text{-model} \otimes \theta = 2\pi)$

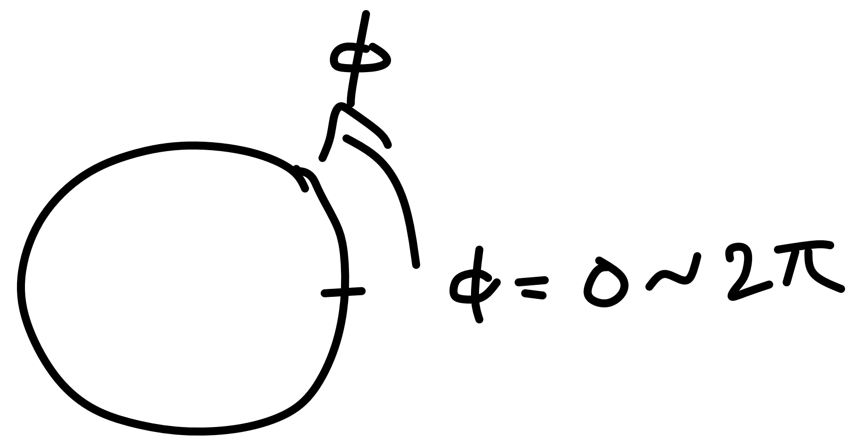


- 4d Yang-Mills theory w/ θ-term $\quad \mathcal{L} = \frac{1}{g^2} \mathrm{tr}(F \wedge * F) + i \frac{\theta}{8\pi^2} \mathrm{tr}(F \wedge F)$



Toy Example : QM for a particle on S^1

$$L = \frac{m}{2}\dot{\phi}^2 - i\frac{\theta}{2\pi}\dot{\phi}$$



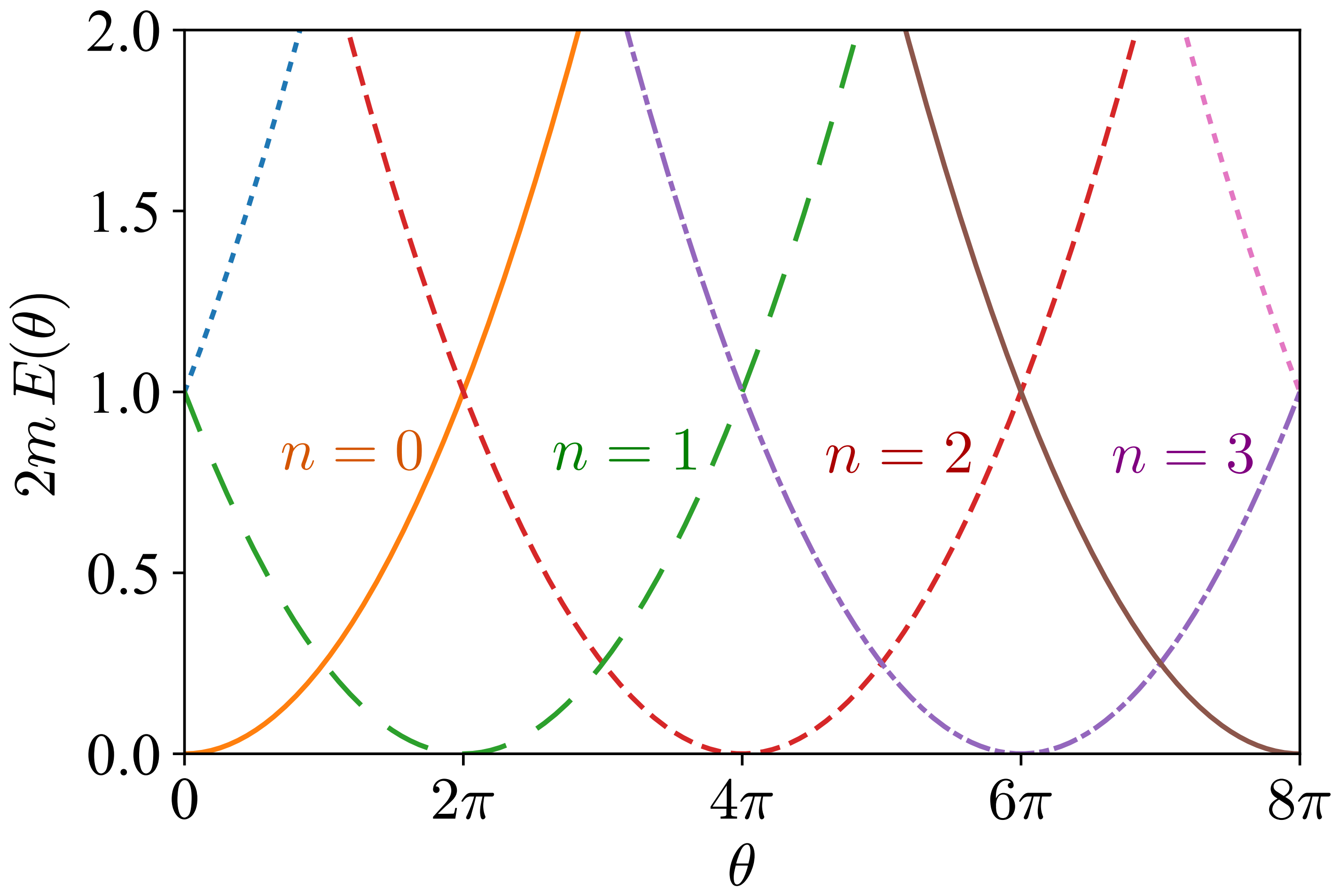
The system has the U(1) symmetry $\phi \mapsto \phi + \alpha$.

Consider the partition function w/ the background U(1) gauge field $A = A_0 d\tau$:

$$Z_\theta[A] = \int \mathcal{D}\phi \, \exp\left(-\int_0^\beta d\tau \left(\frac{m}{2}(\dot{\phi}+A_\tau)^2 - i\frac{\theta}{2\pi}(\dot{\phi}+A_\tau)\right)\right).$$

The θ-periodicity is violated by the background gauge field :

$$Z_{\theta+2\pi}[A] = e^{i\oint A} Z_\theta[A].$$



- level crossing occurs along $\theta \to \theta + 2\pi$.
- U(1) charge n is shifted by 1.

Side remark : Level crossing for ground states

Usually, in QM, level crossing does not occur in ground states.

$\Leftarrow$ This is "forbidden" by the uniqueness of the ground state, and the θ-term gives an exception.

(Proof of uniqueness)

Consider the matrix element of $e^{-\hat{H}}$:

$$\langle x_1 | e^{-\hat{H}} | x_0 \rangle = \int_{x(0)=x_0}^{x(1)=x_1} \mathcal{D}x \; e^{-S[x]} .$$

If $S[x] \in \mathbb{R}$, $\langle x_1 | e^{-\hat{H}} | x_0 \rangle \underset{+}{>} 0$ for any x_0, x_1. $\Rightarrow$ Apply Perron-Frobenius. $\blacksquare$

The above argument cannot be used if $S[x] \in \mathbb{C}$, and the θ-term is indeed a complex phase in the Euclidean path integral.

Thus, though FRG has been tested in many QM models, the study of θ gives a genuinely new test for FRG.

① Obstacles for FRG to study θ : Compactness of ϕ

- $\phi \sim \phi + 2\pi$. How do we encode periodicity of variables in FRG?

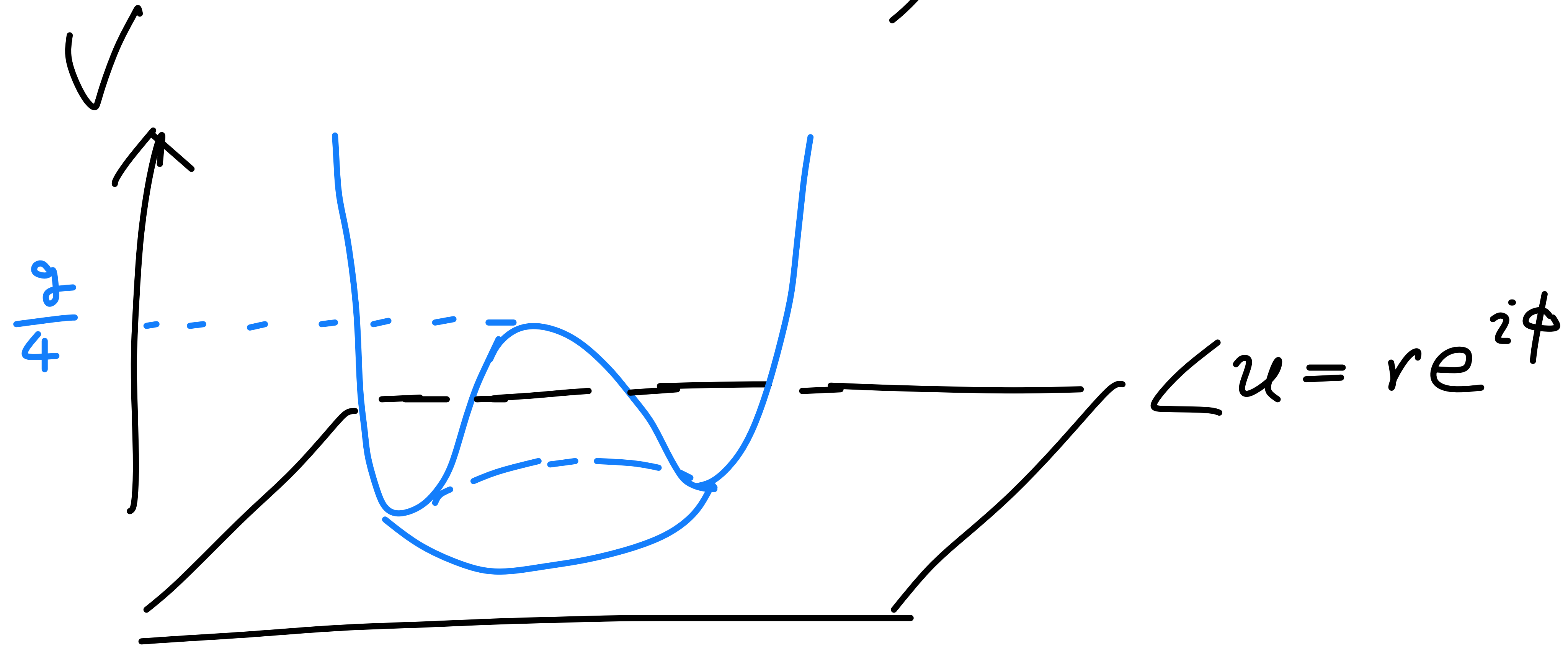
- Topological term $\frac{\theta}{2\pi}\dot{\phi}$ is (almost) total derivative.
  The effect of θ does not appear in the functional differential eq.

(* Despite the above features of θ,
θ affects local physics, e.g. $E(\theta) \sim \frac{1}{2m}\theta^2$.)

<u>Our trick</u> (specific to this example)

$$S^1 \ni e^{i\phi} \xrightarrow{\text{extend}} \mathbb{C} \ni u$$

take $g \to \infty$ limit



$$\mathcal{L} = \frac{m}{2}\dot{u}^*\dot{u} - \frac{\theta}{4\pi}(u^*\dot{u} - \dot{u}^* u) + \frac{g}{4}(u^*u - 1)^2$$

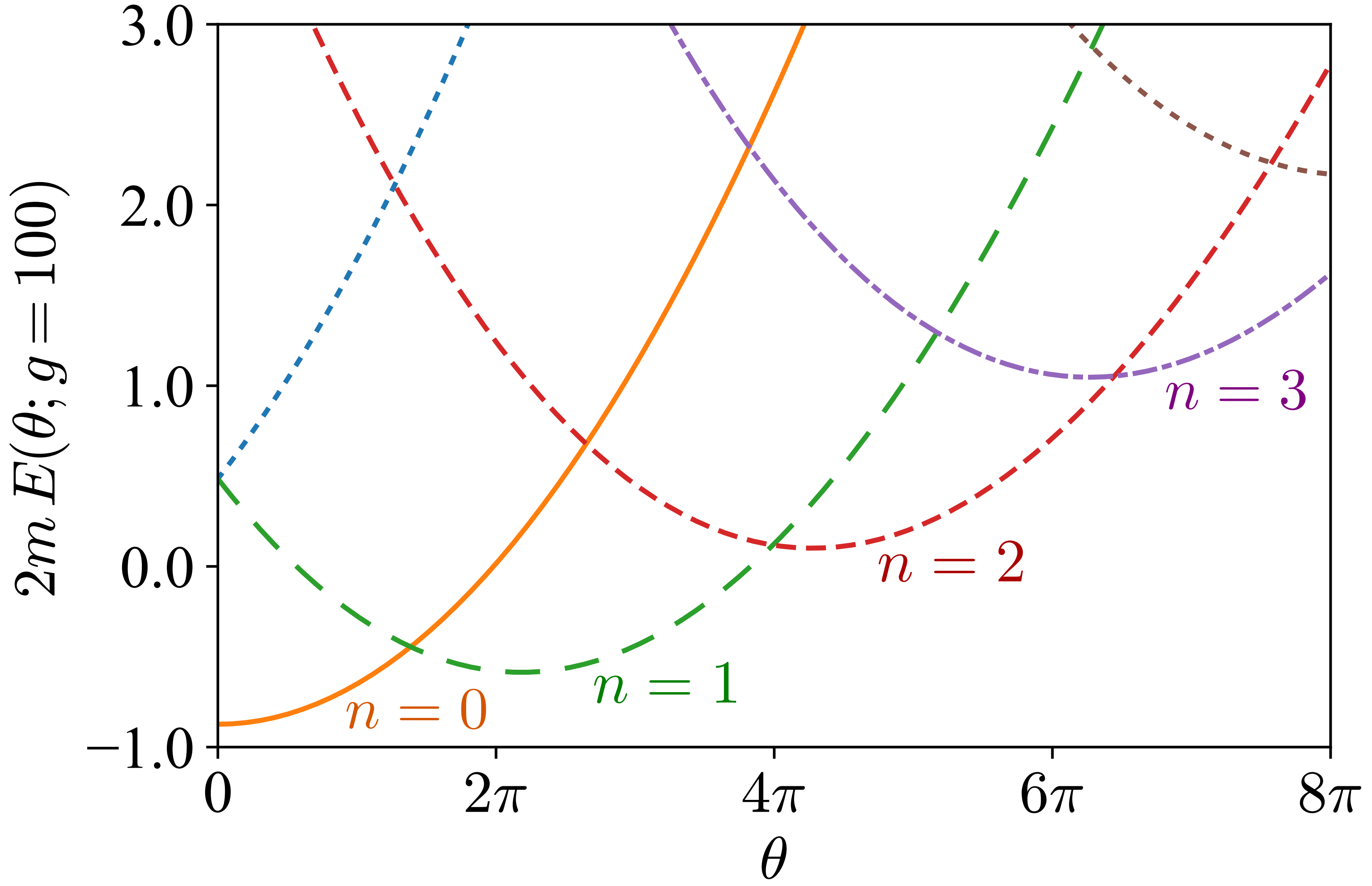
$$\left(= \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - i\frac{\theta}{2\pi}r^2\dot{\phi} + \frac{g}{4}(r^2-1)^2 \right).$$

Properties of QM w/ the wine-bottle potential

$$\mathcal{L} = \frac{m}{2}\dot{u}^*\dot{u} - \frac{\theta}{4\pi}(u^*\dot{u} - \dot{u}^* u) + \frac{g}{4}(u^*u-1)^2$$

$$\left(= \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - i\frac{\theta}{2\pi}r^2\dot{\phi} + \frac{g}{4}(r^2-1)^2 \right).$$

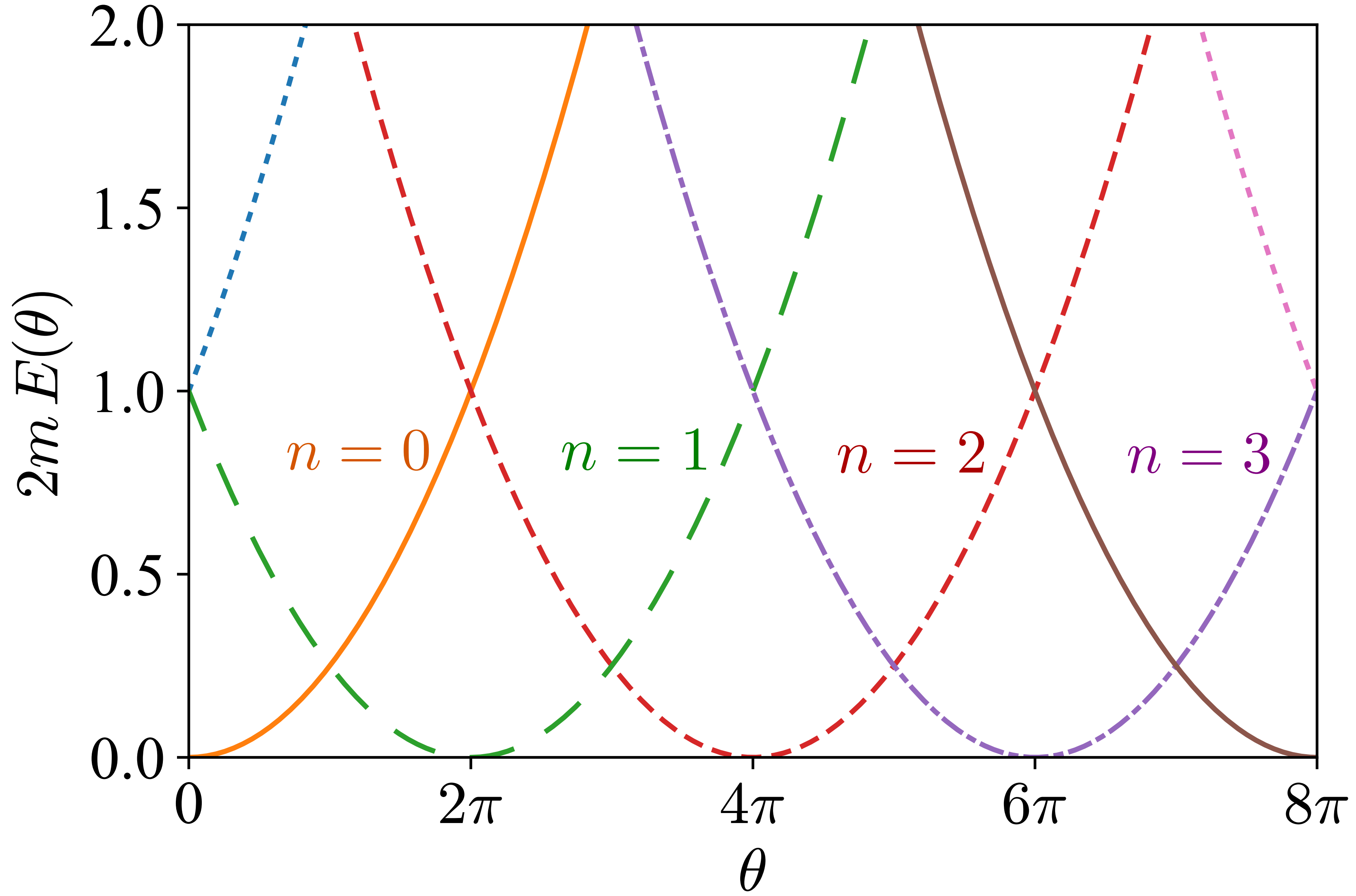
$$\left(\begin{array}{ccc} S^1 & \xleftarrow{g\to\infty} & \mathbb{C} \\ \rotatebox{90}{$\in$} & \underset{\text{extend}}{\rightsquigarrow} & \rotatebox{90}{$\in$} \\ e^{i\phi} & & u \end{array} \right)$$



$g \to \infty$

(+ zero-point energy)

$\longrightarrow$



- Level crossing is captured for finite $g \gg 1$.
- $u \in \mathbb{C} \simeq \mathbb{R}^2$. Nontrivial topology emerges at low energies.

  ↳ θ term is not total derivative at all.

  Obstacles ① are circumvented!

② Obstacle for FRG : Non-convexity of W (or Γ)

To develop FRG for this setup, we first consider the Schwinger functional

$$W[J,J^*] = \ln\left\{\int \mathcal{D}u^*\mathcal{D}u \, \exp\left(-S[u^*,u] + \int (u\cdot J + u^*\cdot J^*)\right)\right\},$$

and then we perform the Legendre transform

$$\Gamma[z,z^*] = z\cdot J + z^*\cdot J^* - W[J,J^*].$$

← Is this really possible?

Legendre transform : Convex func. $\longleftrightarrow$ Convex func.

However, when S is complex-valued (i.e. sign problem exists),

convexity of W is not necessarily ensured.

Indeed, for $\theta \neq 0$, the convexity is generically violated.

(W is complex valued, thus does not accept the notion of convexity)

In our model, $U(1)$ symmetry is unbroken:

$$\langle e^{i\phi} \rangle = 0.$$

$\Rightarrow$ Condition

$$Z = \frac{\delta W}{\delta J} = G \cdot J^* + \mathcal{O}(|J|^3)$$

can be solved recursively (for most values of θ):

$$J = G^{-1} \cdot Z^* + \cdots.$$

In this way, we recursively define the effective action $\Gamma[Z, Z^*]$, and apply FRG to it.

(But, this is just a temporary expedient...
It's an important question if FRG really does not suffer from the sign problem.)

<u>Comment</u>: Polchinski-type FRG does not have this issue.

Application of FRG + LPA

Ansatz

$$\Gamma_k = \int d\tau \left\{ \frac{m}{2}\dot{u}^*\dot{u} - \frac{\theta}{4\pi}\left(u^*\dot{u} - \dot{u}^* u\right) + V_k\left(|u|^2\right) \right\}.$$

w/

$$V_{k=\Lambda} = \frac{g}{4}\left(|u|^2 - 1\right)^2 .$$

We use the Litim regulator

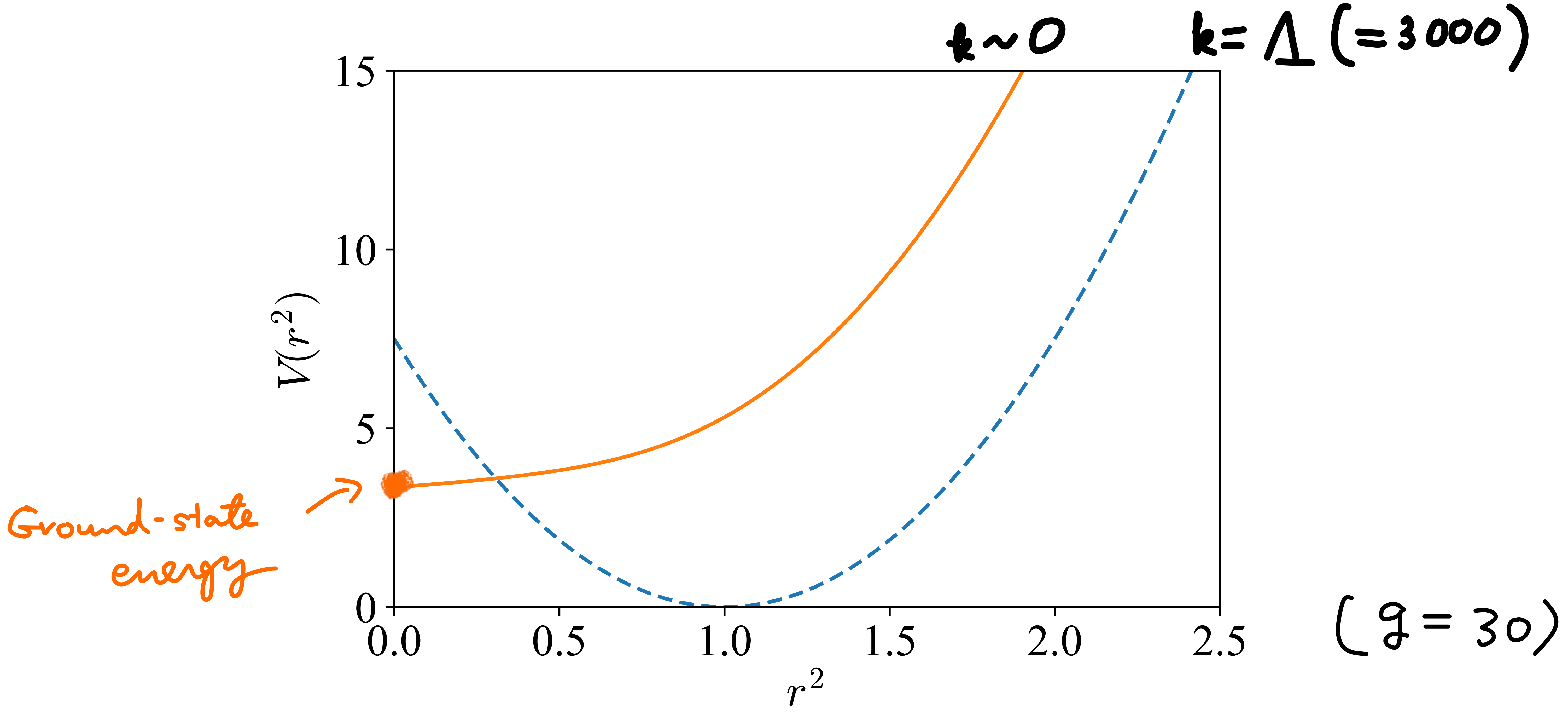
$$R_k(p) = m\left(k^2 - p^2\right)\Theta\left(k^2 - p^2\right),$$

and then the Wetterich eq.

$$\partial_k \Gamma_k = \frac{1}{\Gamma_k^{(2)} + R_k}$$ is solved numerically:

$$\partial_k V_k = \frac{2mk}{|\theta|}\,\frac{mk^2 + 2V' + 2r^2 V''}{\sqrt{\left(mk^2 + 2V' + 2r^2V''\right)^2 - \left(2r^2V''\right)^2}}\,\arctan\left(\frac{k|\theta|}{\pi\sqrt{\cdots}}\right) - \frac{2}{\pi}.$$

RG flow @ $\theta = 0$

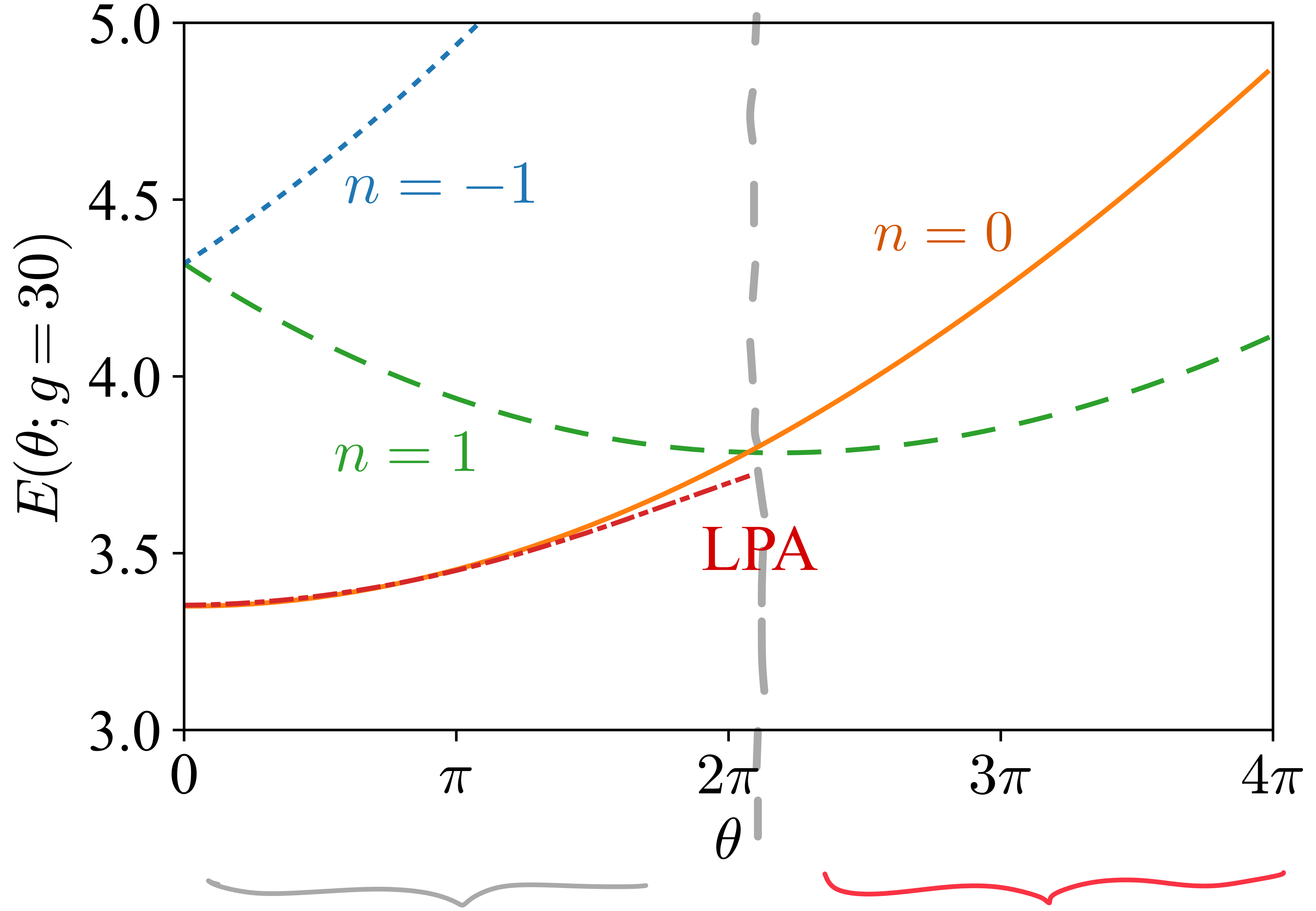


- RG flow restores classically broken U(1) symmetry, and this confirms

$$\langle e^{i\phi} \rangle = 0 .$$

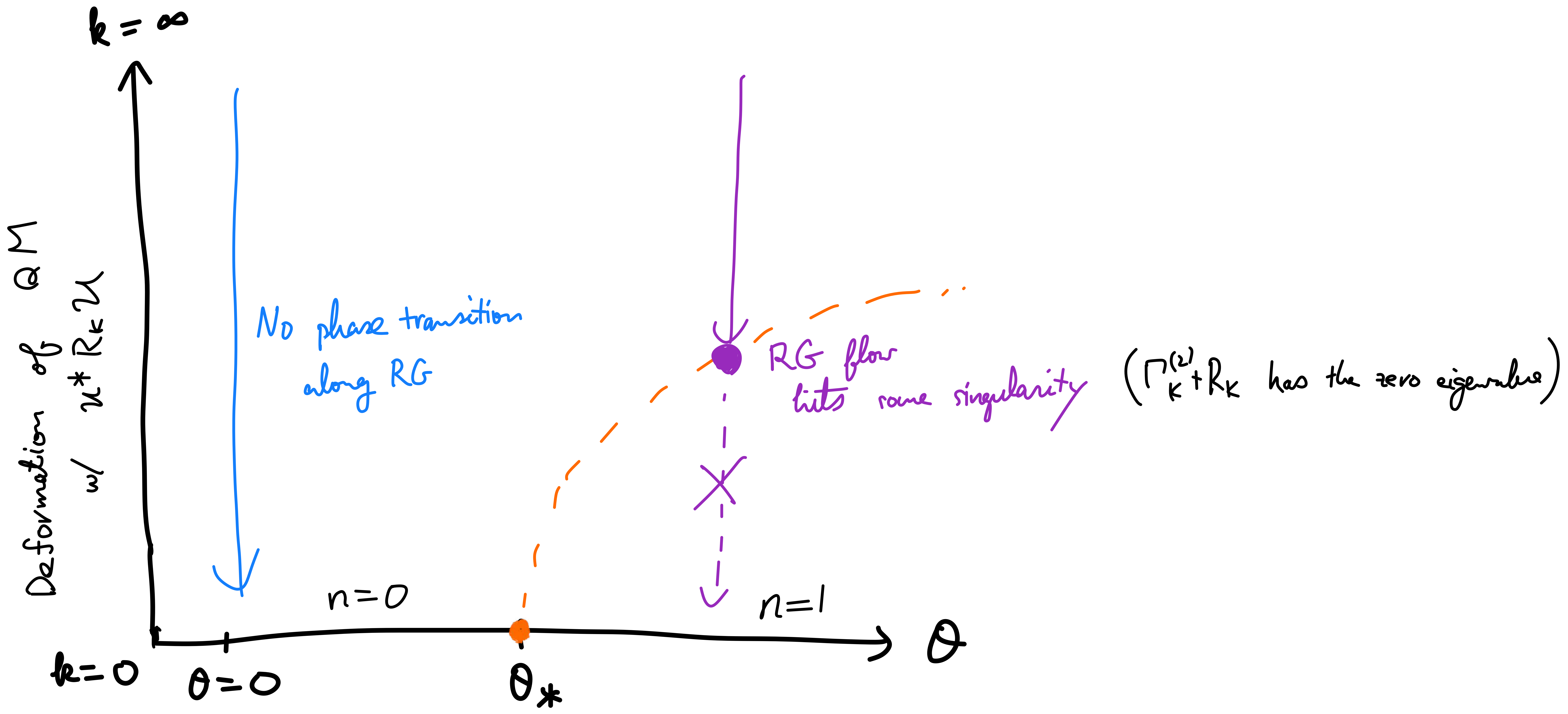
- Ground-state energy matches with that of the Hamiltonian method.

θ - dependence



Until level crossing, FRG+LPA gives a reasonable value for the G.S. energy.

$\partial_{k_*} V_{k_*} = \infty$ for some value of $k = k_* > 0$, and FRG CANNOT be solved.



* Although we have tried only FRG + LPA, the above picture shows that the RG flow cannot circumvent the singularity as long as the symmetry and locality are respected.

Summary

The physics of θ is studied for QM using FRG.

We encountered the following issues at the level of formalisms.

- Compactness of variables $\Rightarrow$ $\begin{array}{c} S^1 \\ \Psi \\ e^{i\phi} \end{array} \underset{\text{extend}}{\overset{\text{deep wine-bottle}}{\rightleftarrows}} \begin{array}{c} \mathbb{C} \\ \Psi \\ u \end{array}$

- Non-convexity of W $\Rightarrow$ Legendre transform is performed "perturbatively".

Until level crossing, FRG + LPA gives a reasonable result,
but the RG flow hits some singularities beyond it.

↳ Do we need some Non-Local treatment of FRG to find SPTs?