Incompleteness of the Large N Analysis of the O(N) Models: Nonperturbative Cuspy Fixed Points and their Nontrivial Homotopy at finite N

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O(N) models

- They have played an important role in our understanding of second order phase transitions.

- N-component vector order parameter
  N=1…Ising, N=2…XY, N=3…Heisenberg Model

- The playground of almost all the theoretical approaches… Exact solution (2d Ising), Renormalization group（d=4-ε, 2+ε expansion), conformal bootstrap

Everything is known about the criticality of O(N) models?
…This is what we want to challenge in this work.

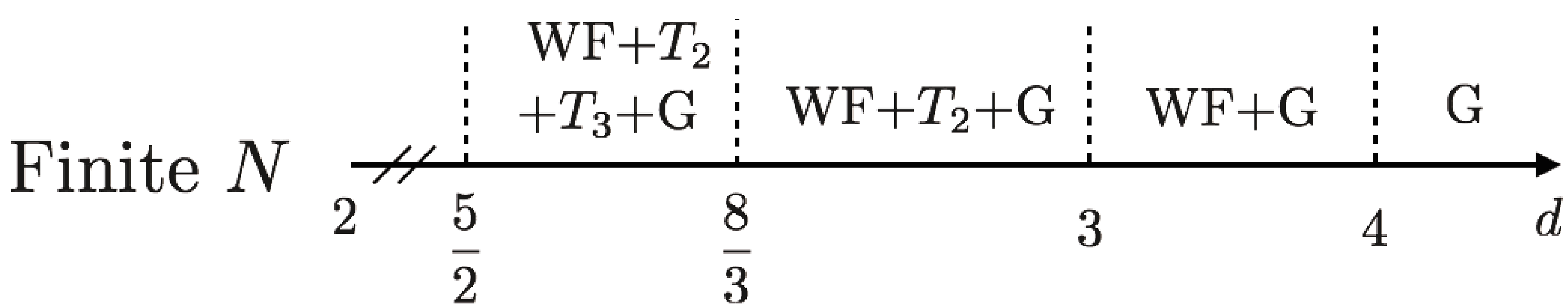
Common wisdom on the criticality of O(N) models (finite N case)

GLW Hamiltonian $$H[\phi] = \frac{1}{2}\int_x (\nabla\phi_i)^2 + U(\phi)$$ ϕ_i

N-component order parameter

$$U(\phi) = a_2\phi_i^2 + a_4(\phi_i^2)^2 + a_6(\phi_i^2)^3 + \dots$$

Below the critical dimension $d_n = 2 + 2/n$, the $(\phi_i^2)^{n+1}$ term becomes relevant around the Gaussian FP (G).



A nontrivial fixed point T_n with n relevant (unstable) directions branches from G at d_n. (Wilson-Fisher FP, which describes second order phase transition, at d=4 and the tricritical FP T_2 at d=3….)

Common wisdom on the criticality of O(N) models at $N = \infty$

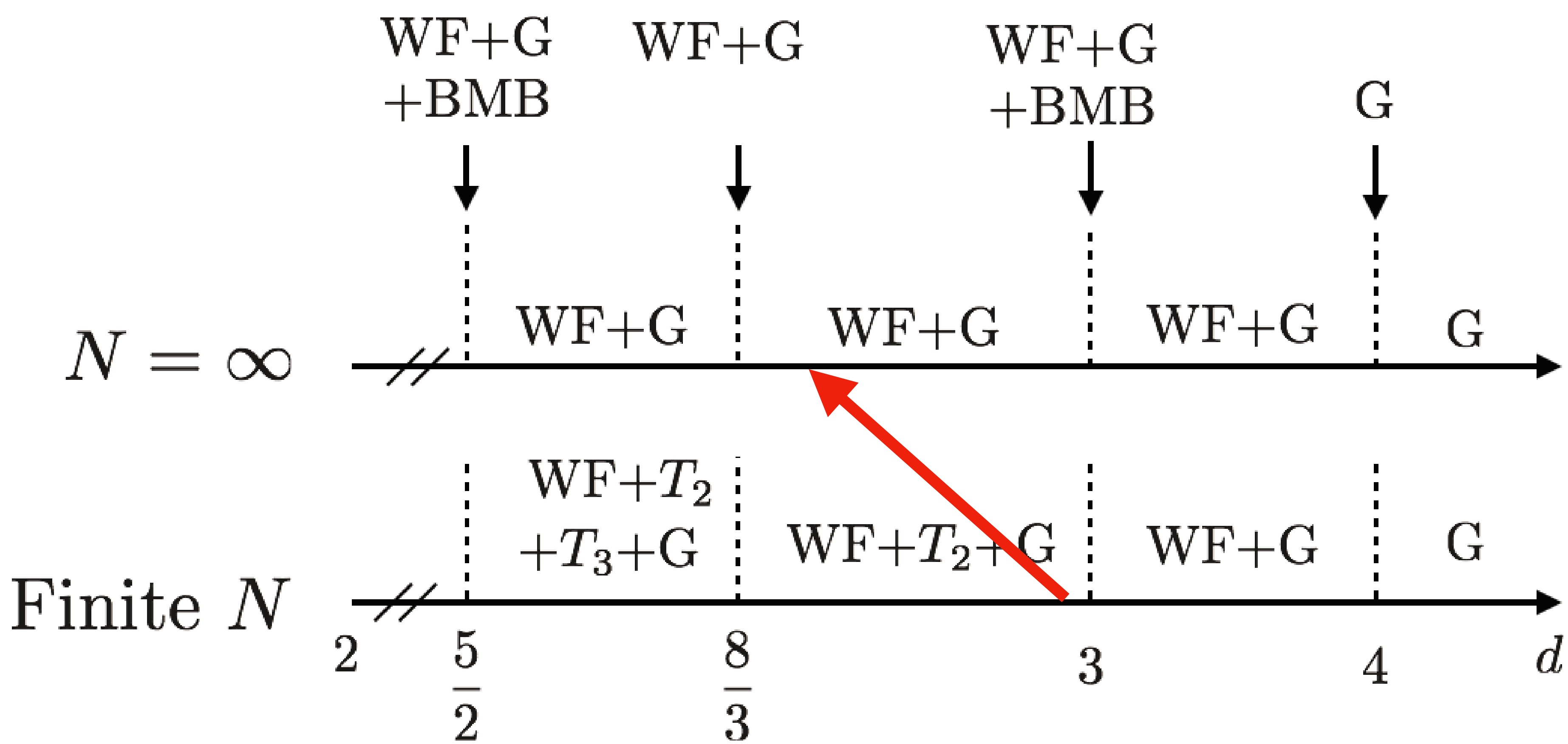
- At $N = \infty$, in generic dimensions 2<d<4, only Gaussian (G) and Wilson-Fisher (WF) FPs have been found.

- Exceptional case: At $d_n = 2 + 2/n$ there exists a line of FPs starting from G. It terminates at BMB (Bardeen-Moshe-Bander) FP for $n = 2,4,6,\ldots$, and at WF FP for odd integer $n = 3,5,7,\ldots$.

  (For the odd integer cases, refer to
  J. Comellas and A. Travesset, Nucl. Phys. B 1997,
  S. Yabunaka and B. Delamotte Arxiv 2301.01021)

- LPA of NPRG is believed to be exact.

Summary of common wisdom and a simple paradox



- What occurs if we follow T2 from $(d = 3^-, N = 1)$ to $(d = 2.8, N = \infty)$ continuously as a function of (d,N)?

Possible scenarios

- T_2 disappears. (Collision with another FP?)
- T_2 becomes singular at $N=\infty$.

Possible scenarios

- T_2 disappears. (Collision with another FP?)
- T_2 becomes singular at $N=\infty$.

We shall see that both possibilities are realized depending on the path followed from $(d = 3^-, N = 1)$ to $(d = 2.8, N = \infty)$, which leads to "nontrivial homopopy" at finite N.

Large-N expansion

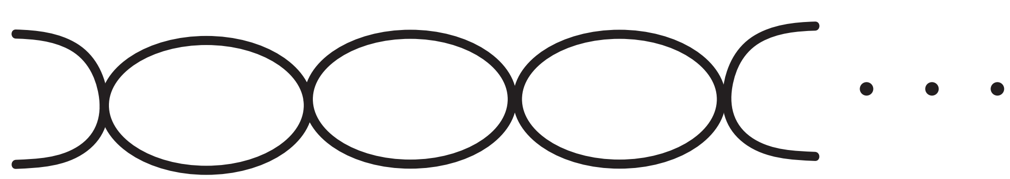
- One of the prominent tools in field theory, which has played an important role in QCD as well as in statistical mechanics and condensed matter physics.

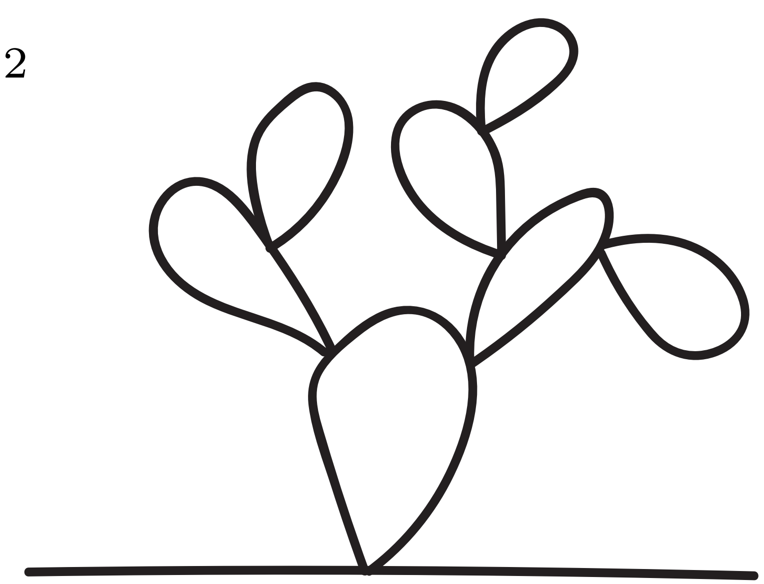
- A nonpertubative method can make a bridge between $d = 4 - \epsilon, 2 + \epsilon$ expansions.

Large-N expansion

- In terms of Feynman graphs, 2 and 4-point functions for O(N) models can be calculated exactly by resumming the bubble and cactus graphs under the assumption $g \sim 1/N$ at the leading order.

g …**coupling constant in front of** $(\varphi^2)^2$





In this talk, the situation can be more complicated than widely believed even for O(N) models.

Usual large N limit of the LPA flow

Rescaled finite N equation $\tilde{U}_t = N\bar{U}_t$ $\tilde{\phi} = \sqrt{N}\bar{\phi}$

$$\partial_t \bar{U}_t(\bar{\phi}) = -d\,\bar{U}_t(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}_t'(\bar{\phi}) + \left(1 - \frac{1}{N}\right)\frac{\bar{\phi}}{\bar{\phi} + \bar{U}_t'(\bar{\phi})} + \frac{1}{N}\frac{1}{1 + \bar{U}_t''(\bar{\phi})}$$

- The terms proportional to 1/N are assumed to be subleading.
- At N=∞, the resulting NPRG eq without an explicit 1/N dependence was believed to be exact and can be solved exactly.

Usual large N limit of the LPA flow

$$\partial_t \bar{U}_t(\bar{\phi}) = -d\,\bar{U}_t(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}_t'(\bar{\phi}) + \left(1 \cancel{- \frac{1}{N}}\right)\frac{\bar{\phi}}{\bar{\phi}+\bar{U}_t'(\bar{\phi})} + \cancel{\frac{1}{N}\frac{1}{1+\bar{U}_t''(\bar{\phi})}}$$

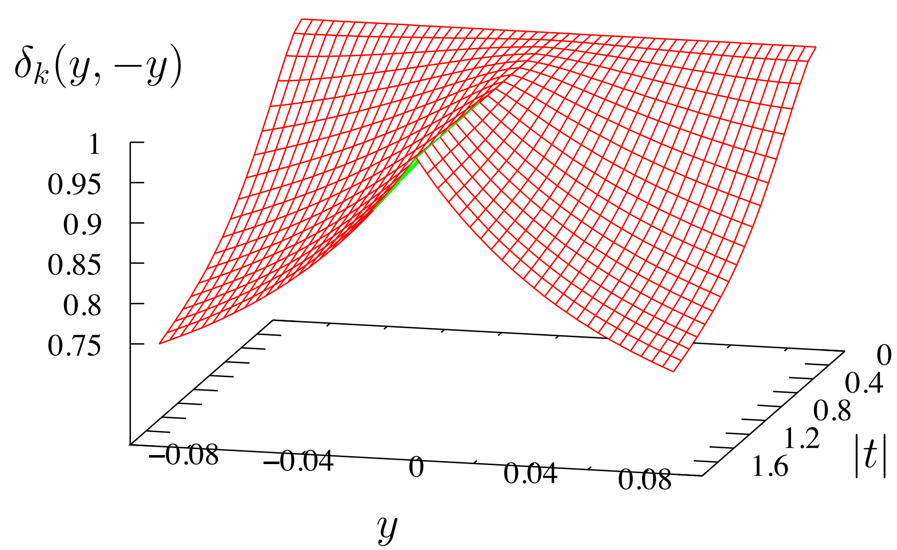
- The only nontrivial solution is Wilson Fisher FP solution in generic dimensions 2<d<4.

- In $d_n = 2 + 2/n \ (n = 2, 3, \cdots)$, we have a line of multicritical FPs starting from the Gaussian FP

- We show that the procedure described here is too restrictive.

Renormalization group FPs showing cusps

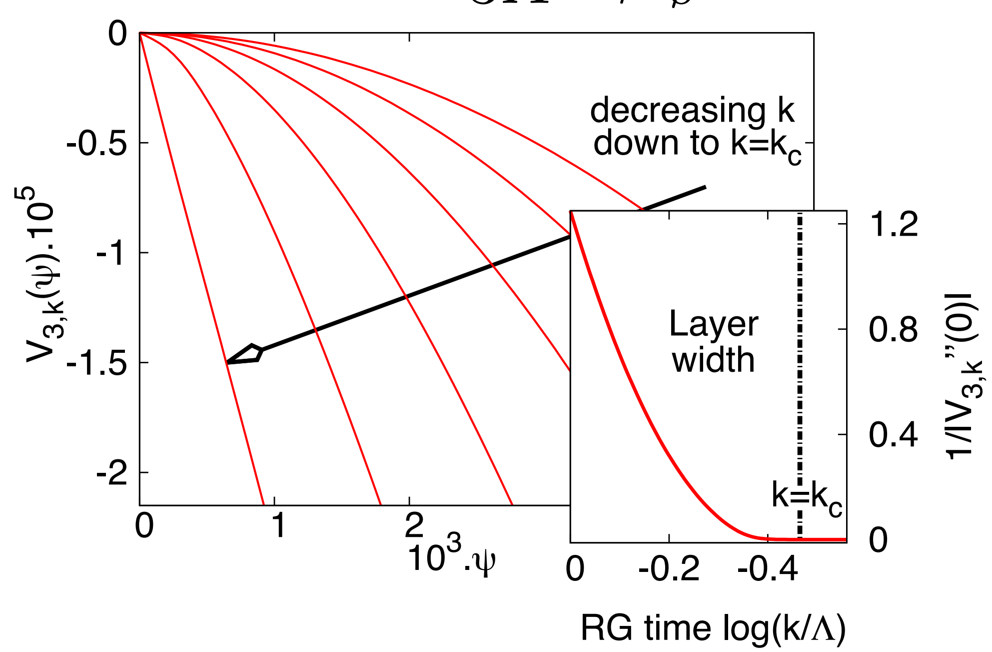
Random field Ising



M. Tissier and G. Tarjus, PRL (2013)

"PCPD"

$$2A \to 3A$$
$$2A \to \emptyset$$
$$3A \to \emptyset$$

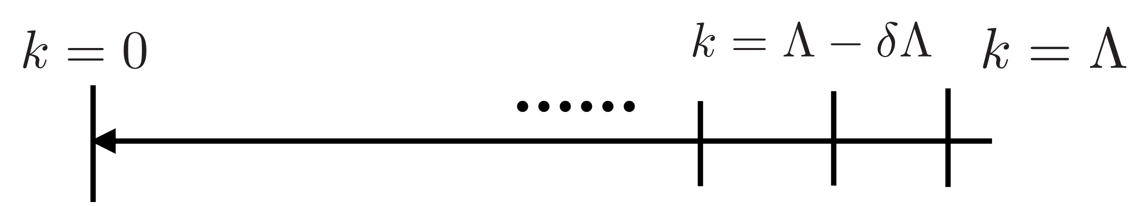


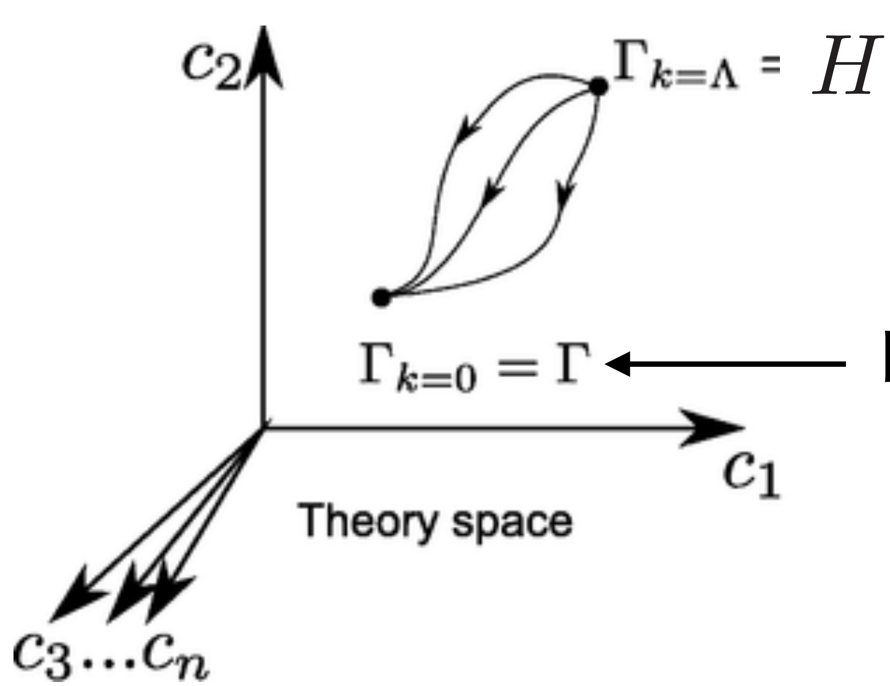
D. Gredat, et al, PRE (2013)

We will show that they also play an important role in simple field theories such as $O(N)$ models.

Non perturbative renormalization group (NPRG)

- Modern implementation of Wilson's RG that takes the fluctuation into account step by step in lowering the cut-off wavenumber k, in terms of wavenumber-dependent effective action Γ_k





Microscopic Hamiltonian

Effective action (Free energy), where all the fluctuations are taken into account.

NPRG equation

NPRG equation (Wetterich, Phys. Lett. B, 1993) is

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}[\partial_t R_k(q^2)(\Gamma_k^{(2)}[q,-q;\phi] + R_k(q))^{-1}]$$

$$t = \ln(k/\Lambda)$$

Derivative expansion(DE2)

- It is impossible to solve the NPRG equation exactly and we have recourse to approximations,

$$\Gamma_k[\boldsymbol{\phi}] = \int_x \left(\frac{1}{2} Z_k(\rho)(\nabla\phi_i)^2 + \frac{1}{4} Y_k(\rho)(\phi_i \nabla \phi_i)^2 \right.$$
$$\left. + U_k(\rho) + O(\nabla^4) \right) .$$

$$\rho = \phi_i \phi_i / 2$$

- Simpler approximations…LPA(η=0), LPA' approximation

$$Y_k(\rho) = 0$$

$$Z_k(\rho) = \bar{Z}_k$$
$$\downarrow$$
$$\eta_t = -\partial_t \log \bar{Z}_k$$

Applications of DE

PHYSICAL REVIEW E **90**, 062105 (2014)

Reexamination of the nonperturbative renormalization-group approach to the Kosterlitz-Thouless transition

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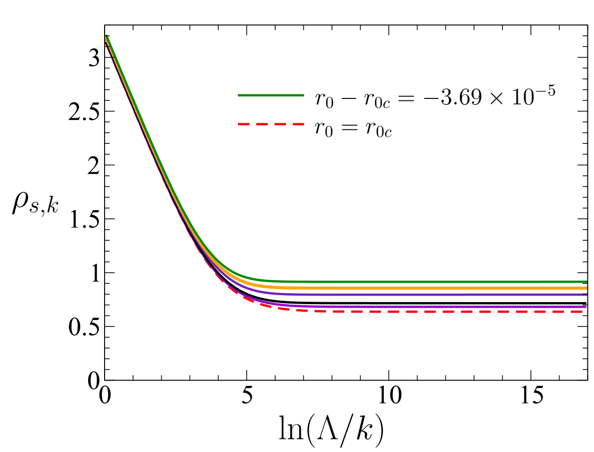
(Received 10 September 2014; published 1 December 2014)

We reexamine the two-dimensional linear O(2) model (φ^4 theory) in the framework of the nonperturbative renormalization-group. From the flow equations obtained in the derivative expansion to second order and with optimization of the infrared regulator, we find a transition between a high-temperature (disordered) phase and a low-temperature phase displaying a line of fixed points and algebraic order. We obtain a picture in agreement with the standard theory of the Kosterlitz-Thouless (KT) transition and reproduce the universal features of the transition. In particular, we find the anomalous dimension $\eta(T_{\rm KT}) \simeq 0.24$ and the stiffness jump $\rho_s(T_{\rm KT}^-) \simeq 0.64$ at the transition temperature $T_{\rm KT}$, in very good agreement with the exact results $\eta(T_{\rm KT}) = 1/4$ and $\rho_s(T_{\rm KT}^-) = 2/\pi$, as well as an essential singularity of the correlation length in the high-temperature phase as $T \to T_{\rm KT}$.

$$\Delta\Gamma_k[\boldsymbol{\phi}] = \frac{1}{2}\rho_{s,k}\int d^d r\,(\nabla\theta)^2.$$



Precision calculation of critical exponents in the O(N) universality classes with the nonperturbative renormalization group

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We compute the critical exponents ν, η and ω of $O(N)$ models for various values of N by implementing the derivative expansion of the nonperturbative renormalization group up to next-to-next-to-leading order [usually denoted $O(\partial^4)$]. We analyze the behavior of this approximation scheme at successive orders and observe an apparent convergence with a small parameter, typically between $\frac{1}{9}$ and $\frac{1}{4}$, compatible with previous studies in the Ising case. This allows us to give well-grounded error bars. We obtain a determination of critical exponents with a precision which is similar or better than those obtained by most field-theoretical techniques. We also reach a better precision than Monte Carlo simulations in some physically relevant situations. In the O(2) case, where there is a long-standing controversy between Monte Carlo estimates and experiments for the specific heat exponent α, our results are compatible with those of Monte Carlo but clearly exclude experimental values.

	ν	η	ω
LPA	0.7090	0	0.672
$O(\partial^2)$	0.6725(52)	0.0410(59)	0.798(34)
$O(\partial^4)$	0.6716(6)	0.0380(13)	0.791(8)
CB (2016)	0.6719(12)	0.0385(7)	0.811(19)
CB (2019)	0.6718(1)	0.03818(4)	0.794(8)
Six-loop, $d = 3$	0.6703(15)	0.0354(25)	0.789(11)
ϵ expansion, ϵ^5	0.6680(35)	0.0380(50)	0.802(18)
ϵ expansion, ϵ^6	0.6690(10)	0.0380(6)	0.804(3)
MC+High T (2006)	0.6717(1)	0.0381(2)	0.785(20)
MC (2019)	0.67169(7)	0.03810(8)	0.789(4)
Helium-4 (2003)	0.6709(1)		
Helium-4 (1984)	0.6717(4)		
XY-AF ($CsMnF_3$)	0.6710(7)		
XY-AF ($SmMnO_3$)	0.6710(3)		
XY-F (Gd_2IFe_2)	0.671(24)	0.034(47)	
XY-F (Gd_2ICo_2)	0.668(24)	0.032(47)	

Scaled NPRG equation

- Fixed point is found by nondimensionalized renormalized field

$$\tilde{\phi} = \sqrt{Z_k} k^{\frac{2-d}{2}} \phi \qquad \tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{U}_t(\tilde{\rho}) = k^{-d} U_k(\rho)$$

Litim cutoff $\quad y = \dfrac{q^2}{k^2} \qquad R_k(q^2) = Z_k k^2 y r(y) \qquad r(y) = (1/y - 1)\theta(1-y)$

Under LPA,

$$\partial_t \tilde{U}_t(\tilde{\phi}) = -d\,\tilde{U}_t(\tilde{\phi}) + \frac{1}{2}(d-2)\tilde{\phi}\,\tilde{U}_t'(\tilde{\phi}) + (N-1)\,\frac{\tilde{\phi}}{\tilde{\phi} + \tilde{U}_t'(\tilde{\phi})} + \frac{1}{1 + \tilde{U}_t''(\tilde{\phi})}.$$

Rescaled finite N equation $\quad \tilde{U}_t = N\bar{U}_t \qquad \tilde{\phi} = \sqrt{N}\bar{\phi}$

$$\partial_t \bar{U}_t(\bar{\phi}) = -d\,\bar{U}_t(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}_t'(\bar{\phi}) + \left(1 - \frac{1}{N}\right)\frac{\bar{\phi}}{\bar{\phi} + \bar{U}_t'(\bar{\phi})} + \frac{1}{N}\frac{1}{1 + \bar{U}_t''(\bar{\phi})}$$

Nondimensionalized NPRG eq.

- Scaling solutions can be found as FPs solution of **nondimensionalized** NPRG eq.

$$\tilde{\phi} = \sqrt{Z_k} k^{\frac{2-d}{2}} \phi \qquad \tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{U}_t(\tilde{\rho}) = k^{-d} U_k(\rho)$$

Litim cutoff $y = \dfrac{q^2}{k^2}$ $\quad R_k(q^2) = Z_k k^2 y r(y)$ $\quad r(y) = (1/y - 1)\theta(1-y)$

Under LPA,

$$\partial_t \tilde{U}_t(\tilde{\phi}) = -d\,\tilde{U}_t(\tilde{\phi}) + \frac{1}{2}(d-2)\tilde{\phi}\,\tilde{U}_t'(\tilde{\phi}) + (N-1)\,\frac{\tilde{\phi}}{\tilde{\phi} + \tilde{U}_t'(\tilde{\phi})} + \frac{1}{1 + \tilde{U}_t''(\tilde{\phi})}.$$

Wilson-Polchinski version of NPRG

Transformation of the variables

$$(U,\phi) \longleftrightarrow (V,\Phi)$$

$$V(\mu) = U(\phi) + (\phi - \Phi)^2/2$$

$$\phi - \Phi = -2\Phi V'(\mu)$$

$$\mu = \Phi^2$$

Rescaling in N $\bar{\mu} = \mu/N,\ \bar{V} = V/N$

LPA FP eq.

$$0 = 1 - d\,\bar{V} + (d-2)\bar{\mu}\bar{V}' + 4\bar{\mu}\bar{V}'^2 - 2\bar{V}' - \underline{\frac{4}{N}\bar{\mu}\,\bar{V}''}.$$

$1/N$	A small parameter
$\bar{V}''$	The highest order derivative

We have to deal with singular perturbation in general.

Usual large-N limit in the functional RG

$$0 = 1 - d\,\bar{V} + (d-2)\bar{\varrho}\bar{V}' + 2\bar{\varrho}\bar{V}'^2 - \bar{V}' - \cancel{\frac{2}{N}\bar{\varrho}\,\bar{V}''}$$

- In generic dimensions $2 < d < 4$, it has three solutions: Gaussian FP (G), Wilson Fisher FP (WF) and linear FP $\bar{V}(\bar{\varrho}) = \bar{\varrho}$.
- In dimensions $d = 2 + 2/p$ with odd integer $p > 0$, $(\boldsymbol{\varphi}^2)^{p+1}$ term is marginal around G and a line of FPs starting from G and terminating at BMB FP appears.

Tricritical FP solutions in $d = 3$ and at $N = \infty$ in LPA

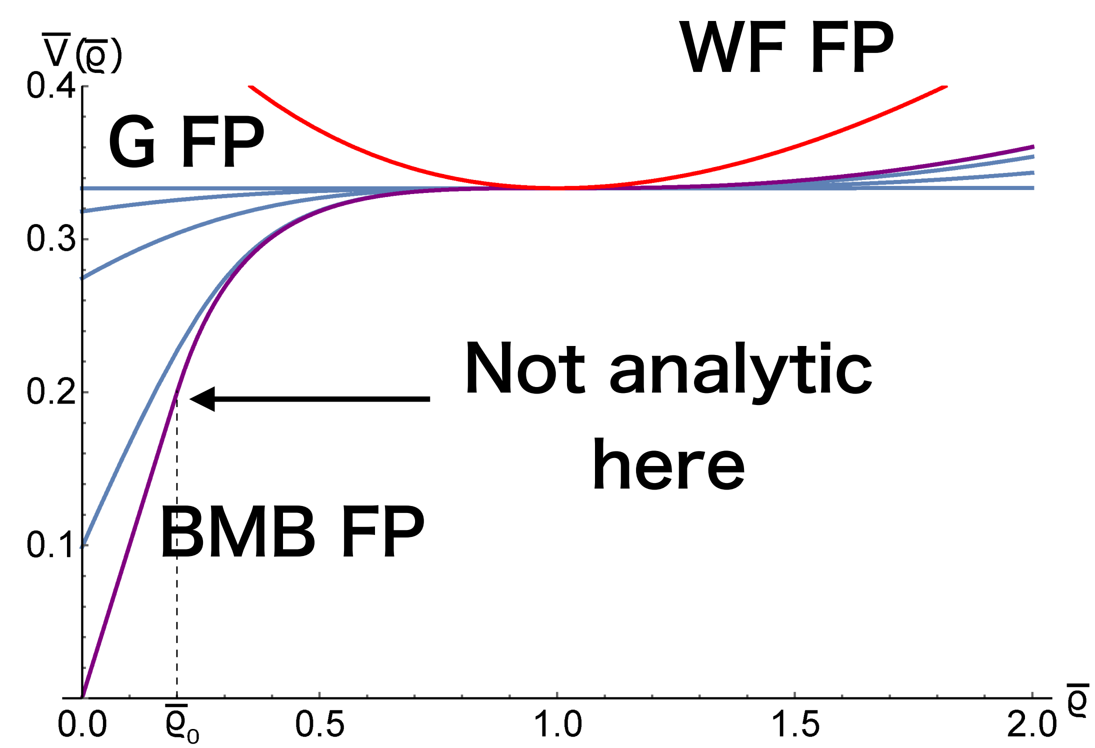
$$\bar{\varrho} = \bar{\mu}/2$$

$$\bar{\varrho}_{\pm} = 1 + \frac{\bar{V}'\left(\frac{5}{2} - \bar{V}'\right)}{\left(1 - \bar{V}'\right)^2} + \frac{\frac{3}{2}\arcsin\sqrt{\bar{V}'} \pm \sqrt{2/\tau}}{\left(\bar{V}'\right)^{-1/2}\left(1 - \bar{V}'\right)^{5/2}}$$

$\bar{\varrho}_+ \dashrightarrow \bar{\varrho} > 1$

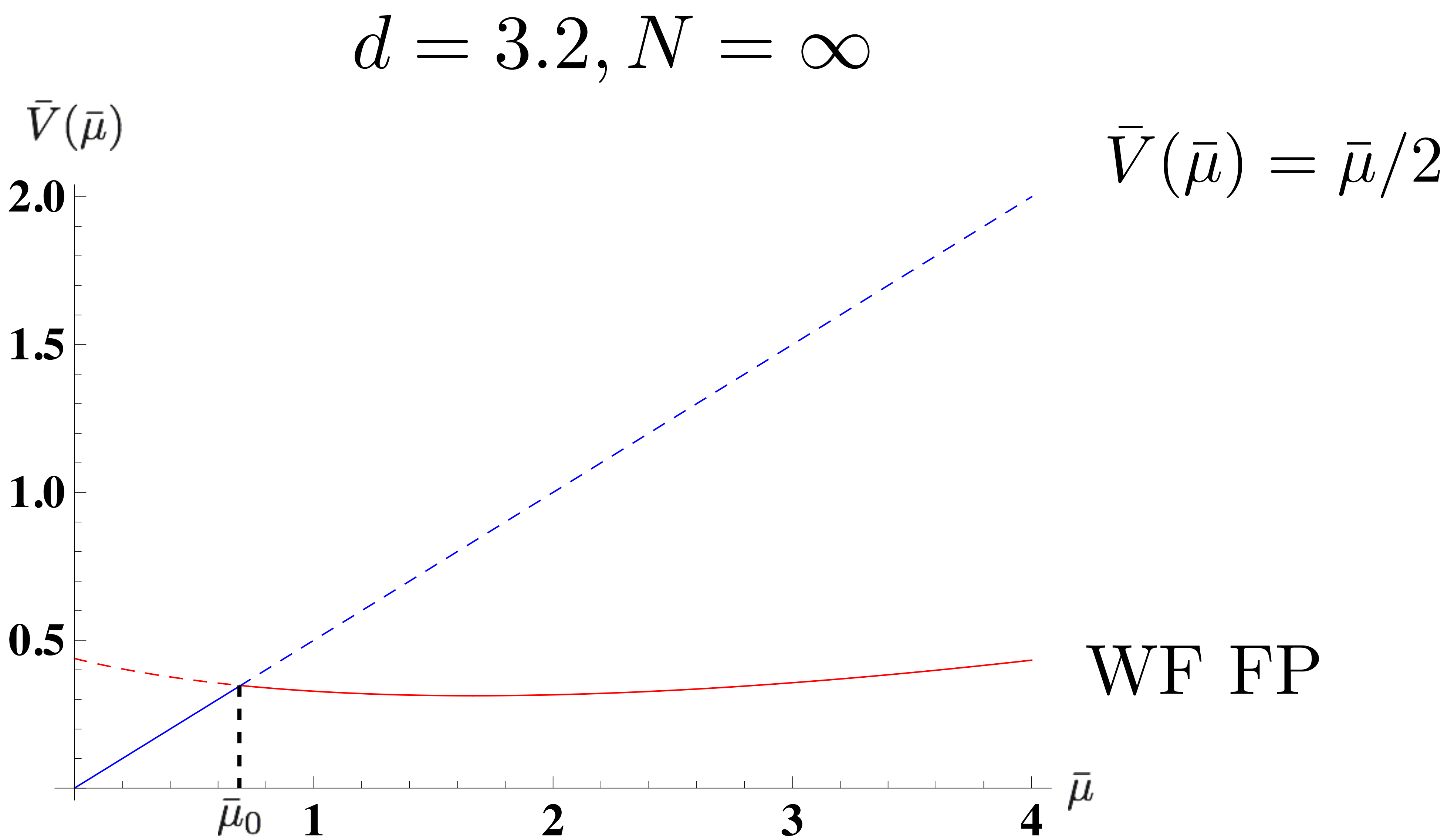
$\bar{\varrho}_- \dashrightarrow \bar{\varrho} < 1$

D. F. Litim and M. J. Trott, PRD (2018)



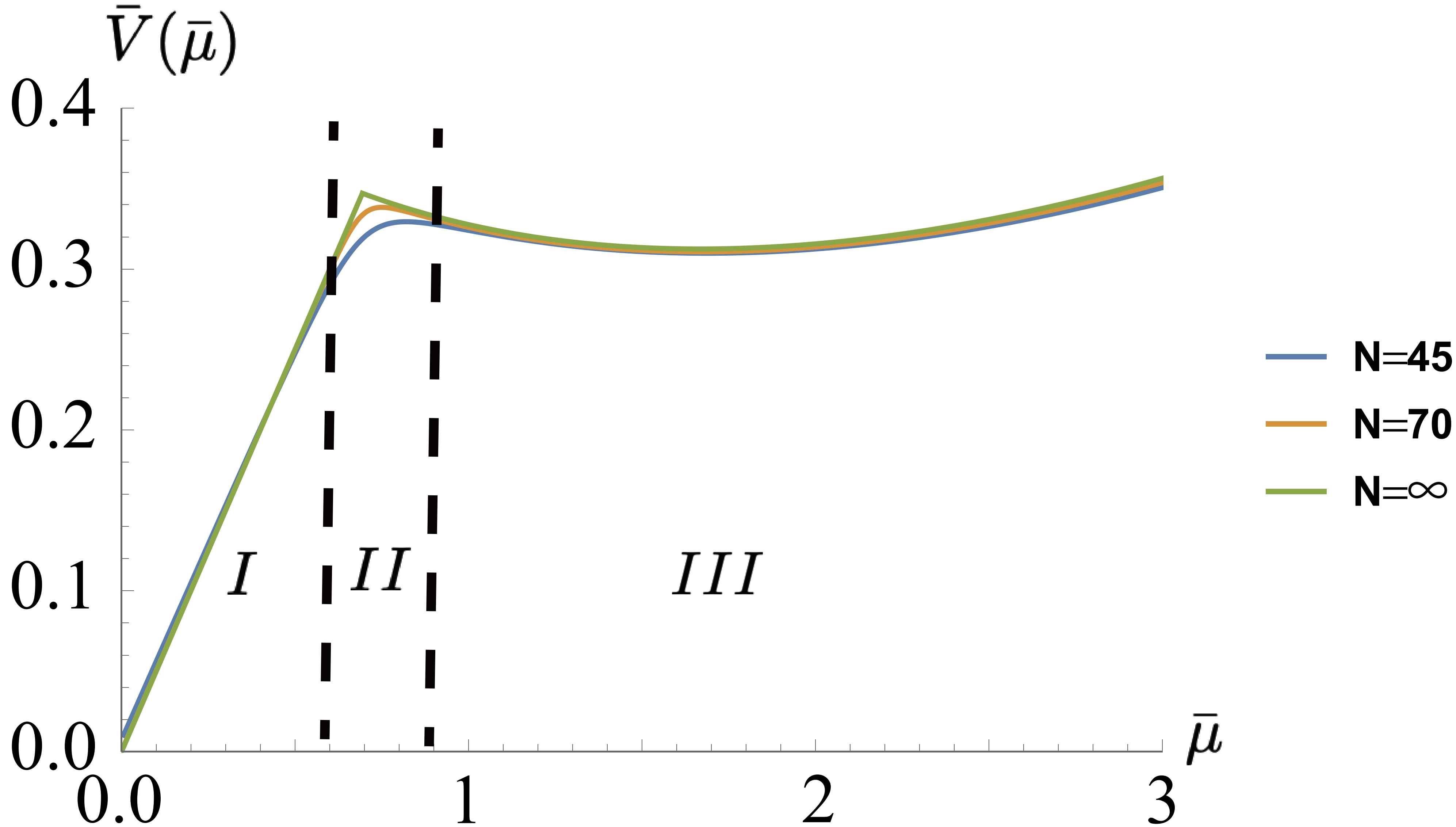
- $\tau = 0$ ⋯Gaussian (G) FP
- $\tau \in [0, \tau_{\mathrm{BMB}} = 32/(3\pi)^2]$ ⋯ FPs on the BMB line
- $\tau > \tau_{\mathrm{BMB}}$ ⋯No FP defined for all ϱ
- $\sqrt{2/\tau} = 0$ ⋯Wilson-Fisher (WF) FP

A FP with a cusp at N=∞



Two smooth FP solutions at N=∞ can be connected with a cusp.

C_2 FP (with 2 relevant directions)



Boundary layer analysis (finite N cases)

$\tilde{\mu} = N(\bar{\mu} - \bar{\mu}_0)$ **Scaled variable around a cusp**

- At the leading order in $1/N$ $F(\tilde{\mu}) = \bar{V}'(\bar{\mu})$

$$0 = 1 - d\,\bar{V}(\bar{\mu}_0) + (d-2)\bar{\mu}_0 F + 4\bar{\mu}_0 F^2 - 2F - 4\bar{\mu}_0\, F'$$

Primes stand for derivatives with respect to $\tilde{\mu}$

- The boundary layer solution near the cusp is given as

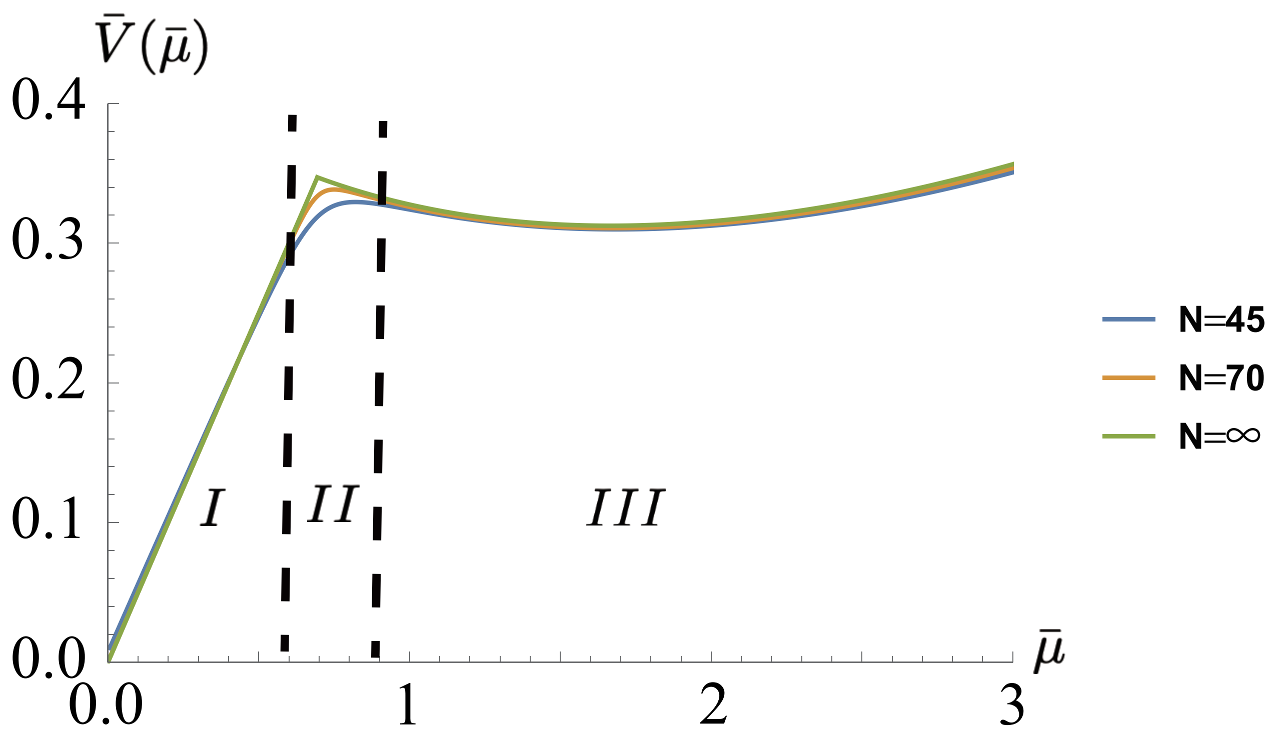
$$V'(\tilde{\mu}) = V_1 - V_2 \tanh(V_2 \tilde{\mu})$$

$$V_1 = 1/4 + \bar{V}'(\bar{\mu}_0^+)/2 \qquad V_2 = 1/4 - V'(\mu_0^+)/2$$

At finite N, the boundary layer matches smoothly (but abruptly) the two different slopes V_1 and V_2 on the right and left of the cusp.

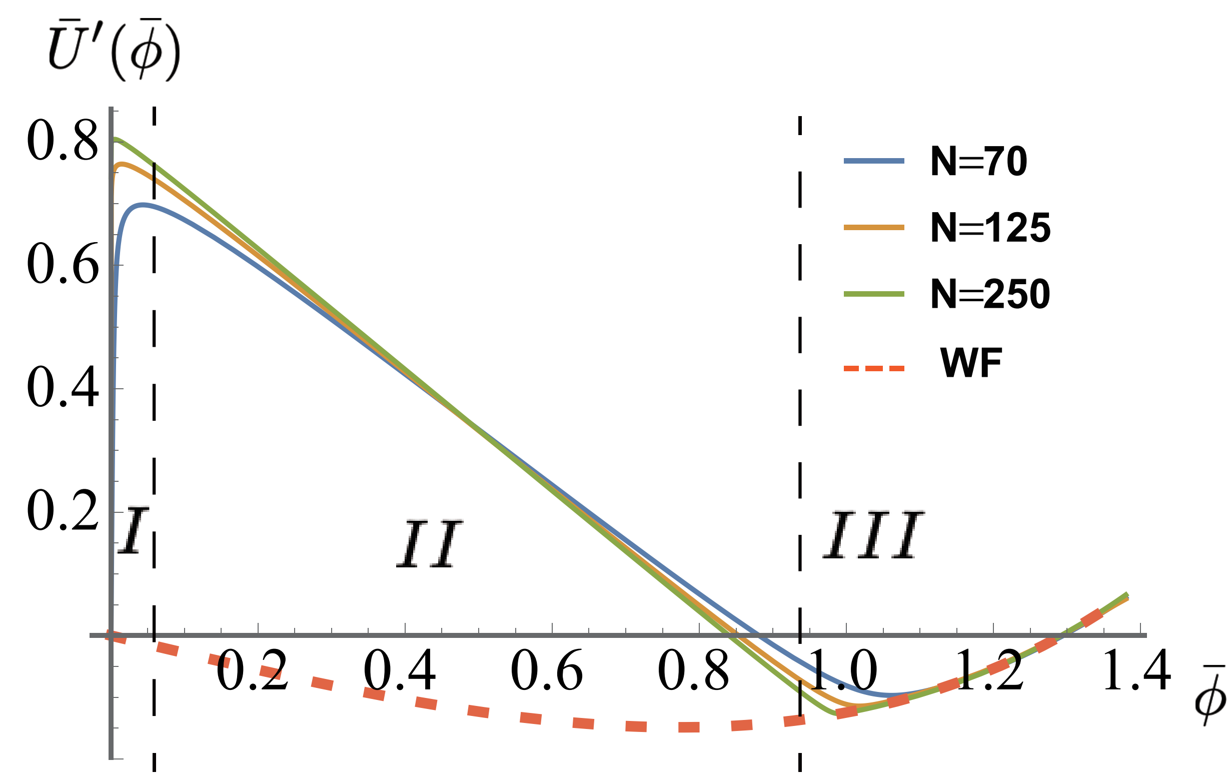
Correspondence between the two parametrization

Wilson-Polchinski framework



$\mu = \Phi^2$

Wetterich framework



$$(U, \phi) \longleftrightarrow (V, \Phi)$$

$$V(\mu) = U(\phi) + (\phi - \Phi)^2/2$$

$$\phi - \Phi = -2\Phi V'(\mu)$$

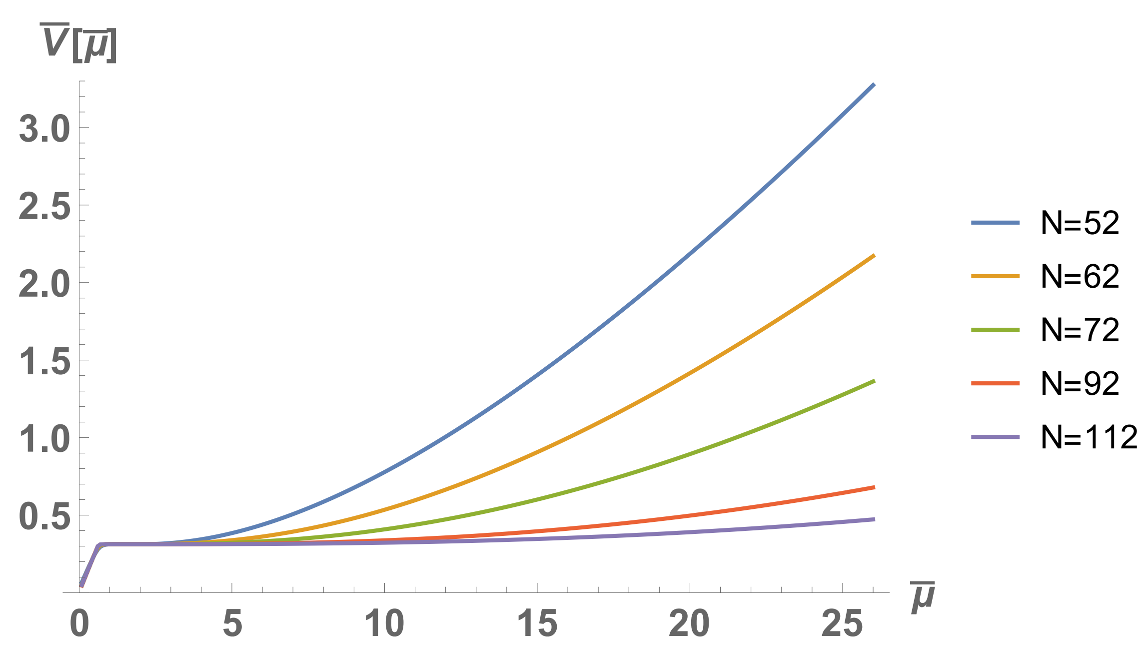
$$\Phi = \phi + \bar{U}'(\bar{\phi})$$

C_3 (with 3 relevant directions)

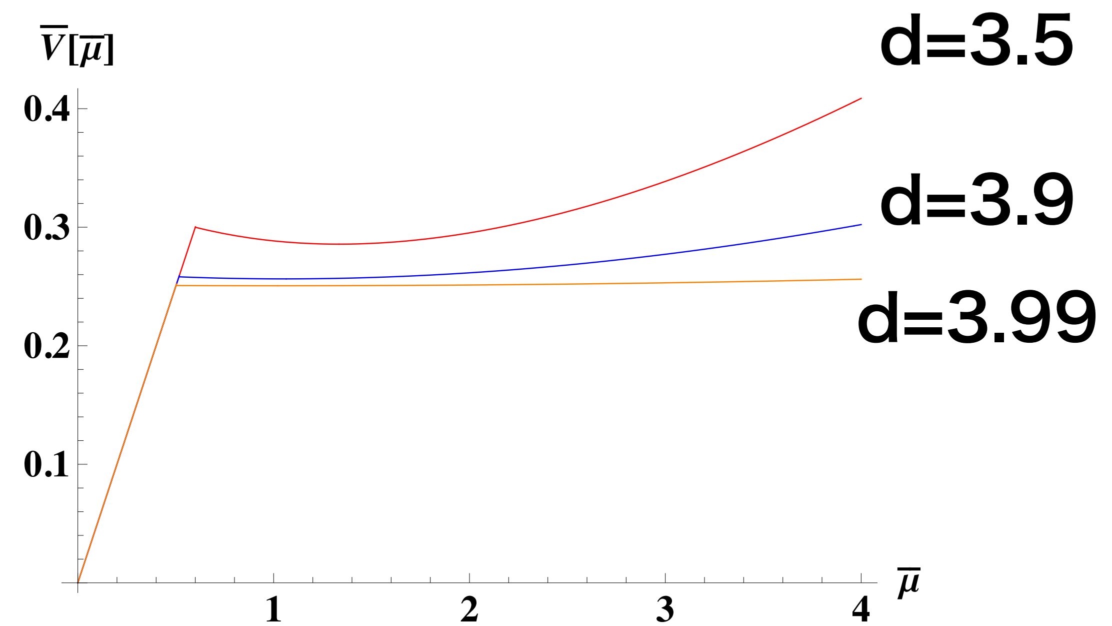
- C_2 is the only singular FP of $O(N)$ models at $N=\infty$??
- We have found that, at $N=\infty$, C_2 and another new FP C_3 appear as a pair in d=4.

Wilson-Polchinski framework

C_3 d=3.2

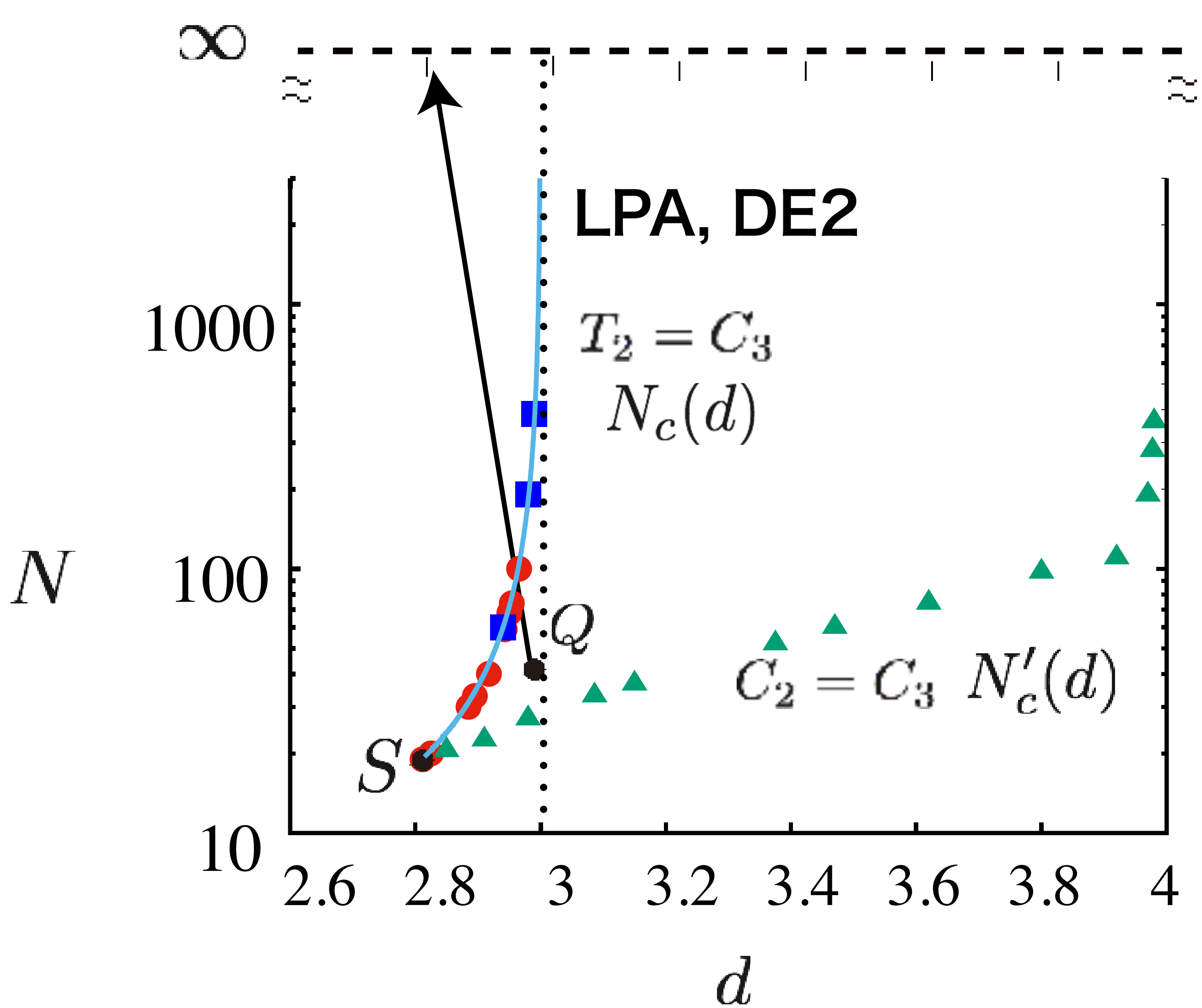


C_2 $N=\infty$



Finite-N fixed point structure

We found two nonperturbative fixed points C_2 (two-unstable) and C_3 (three-unstable), which do not coincide with G at any d.



$$N = N_c(d)$$

T_2 and C_3 collide and vanish

$$N = N'_c(d)$$

C_2 and C_3 collide and vanish

The two lines meet at S=(d=2.8,N=19)

The line $N = N_c(d)$

- We can fit this line as $N_c(d) = 3.6/(3-d)$

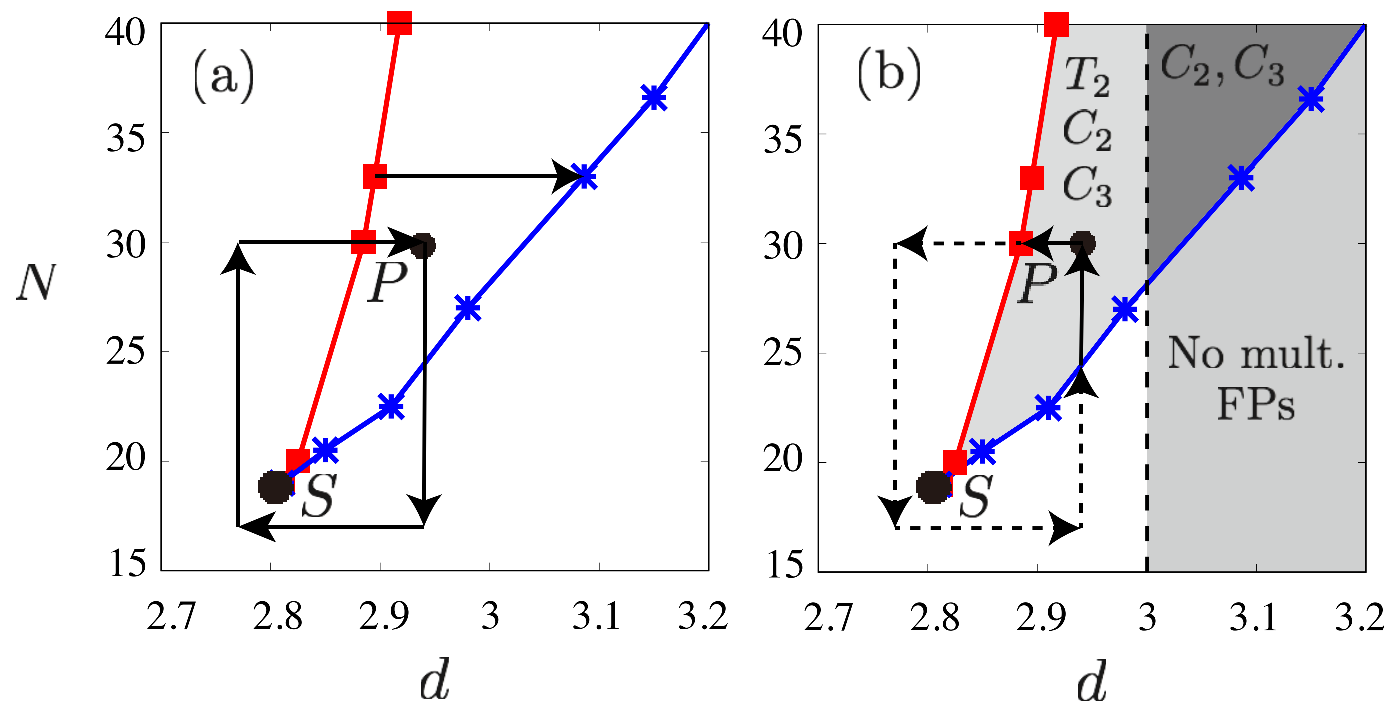
  Pisarski (1982 PRL) and Osborn-Stergiou (2018 JHEP) studied ϕ^6 theory perturbatively (with $\epsilon = 3 - d$ expansion) and showed that T_2 can exist for

$$N \leq N_c^{PT}(d) = \frac{36}{\pi^2(3-d)} \simeq \frac{3.65}{(3-d)}$$

which agrees with our numerical fit within numerical uncertainty.

- The perturbative calculation does not describe the nonperturbative FP C_3 far from the line $N = N_c(d)$ very precisely.

The first double-valued structure

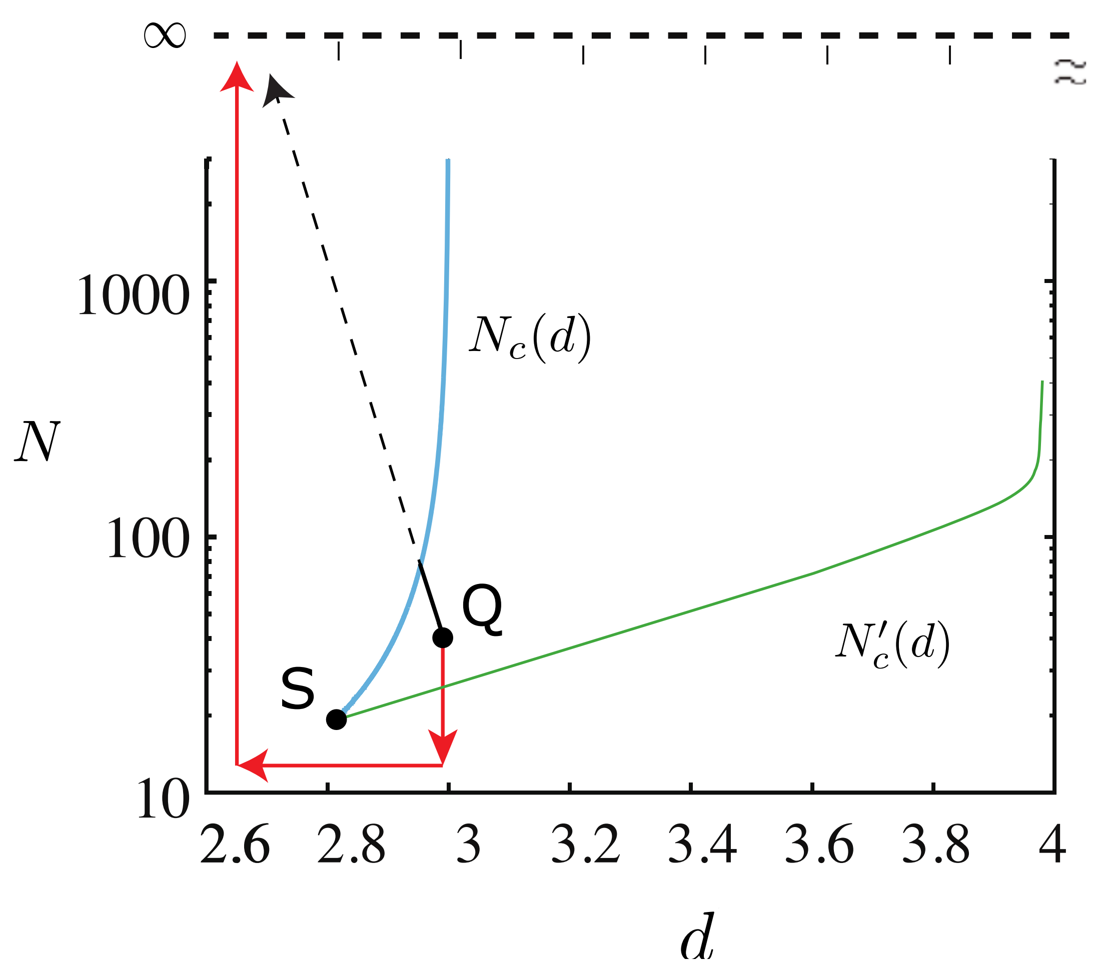


- Starting from P, we follow T_2 around a path around the point S clockwise. After full rotation it becomes C_2 .

- Anticlockwise path… T_2 vanishes at $N = N_c(d)$ and it remains complex all along the dashed path. It becomes real at $N = N_c'(d)$ and comes back as C_2 .

After two full rotations we go back to T_2

Two fates of depending on the path followed



The first double-valued structure

- However complicated it may seem, this structure is one of the simplest ones consistent with the following well known facts:

  (1) Pertubative FP T_2 vanishes above the line $N = N_c(d)$ by colliding another FP C_3
  (2) C_2 and C_3 do not exist in $d > 4$ (Triviality)
  (3) We have T_2 in $d = 2$ and $N = 1$
  (Conformal field theory).

  To obtain the complete FP structure, we need to consider the BMB line.

Singular FPs constructed from the FPs on the BMB line

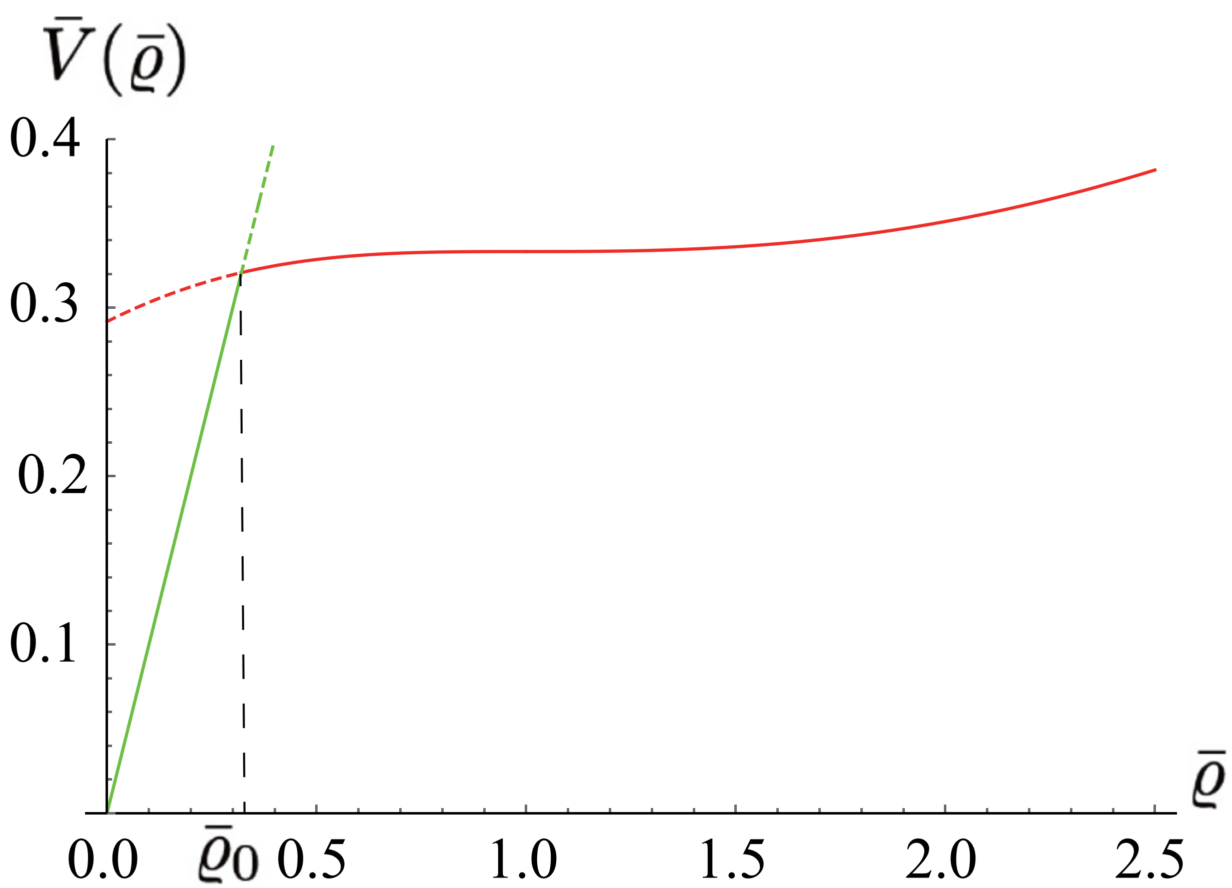
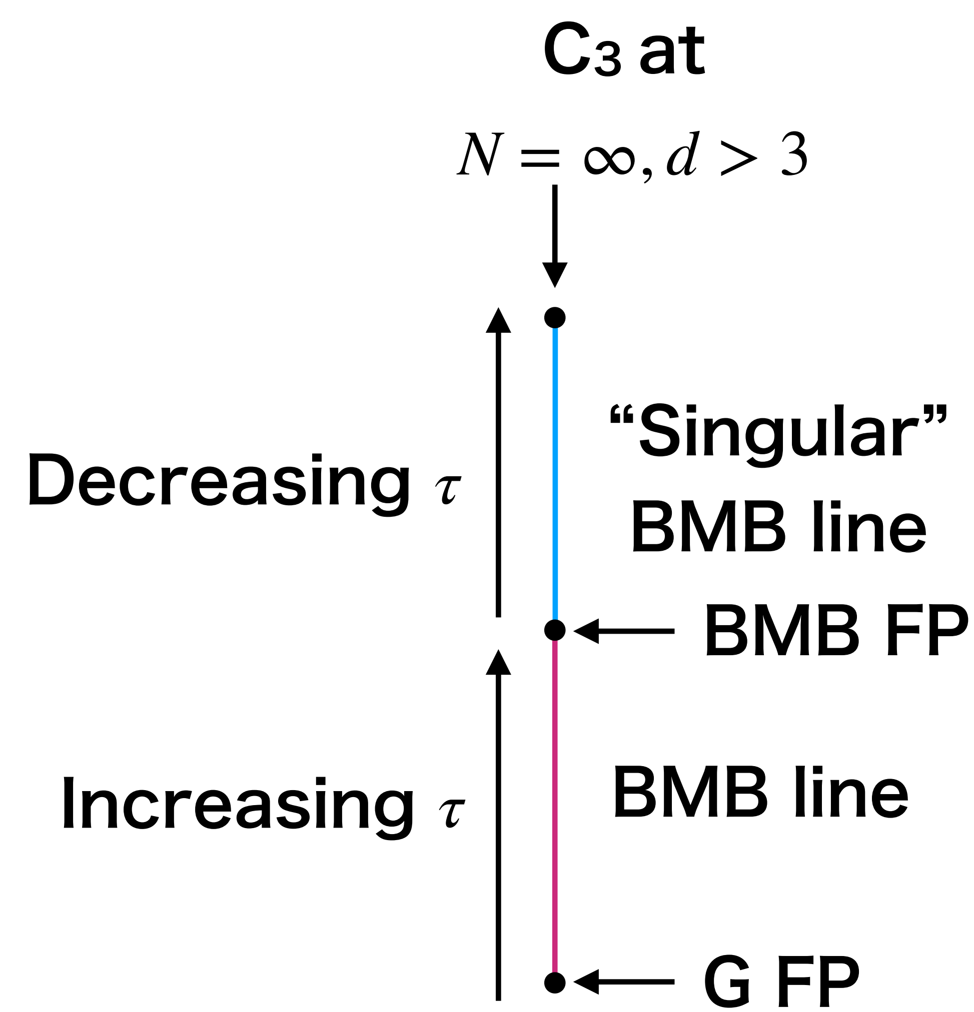


FIG. 7. $d = 3$ and $N = \infty$: Construction of the singular counterpart $S\mathcal{A}$ of a regular tricritical FP $\mathcal{A}$ of the BMB line. The potential of $\mathcal{A}$ is the solution of Eq. (21) obtained with $\tau \simeq 0.1776$ (shown in red). The potential of $S\mathcal{A}$ is shown as a solid line made of two parts that match at $\bar{\varrho}_0 \simeq 0.32$. For $\bar{\varrho} > \bar{\varrho}_0$ it coincides with $\mathcal{A}(\tau)$ and for $\bar{\varrho} < \bar{\varrho}_0$ it is $\bar{V}(\bar{\varrho}) = \bar{\varrho}$.



Finite-N realization of the regular BMB line

- Let us consider to follow T_2 or C_3 on a path toward $(d = 3, N = \infty)$: $d = 3 - \alpha/N\,, N \to \infty$

- It approaches a FP on the BMB line and τ is given by
$$\alpha - 36\tau + 96\tau^2 = 0$$

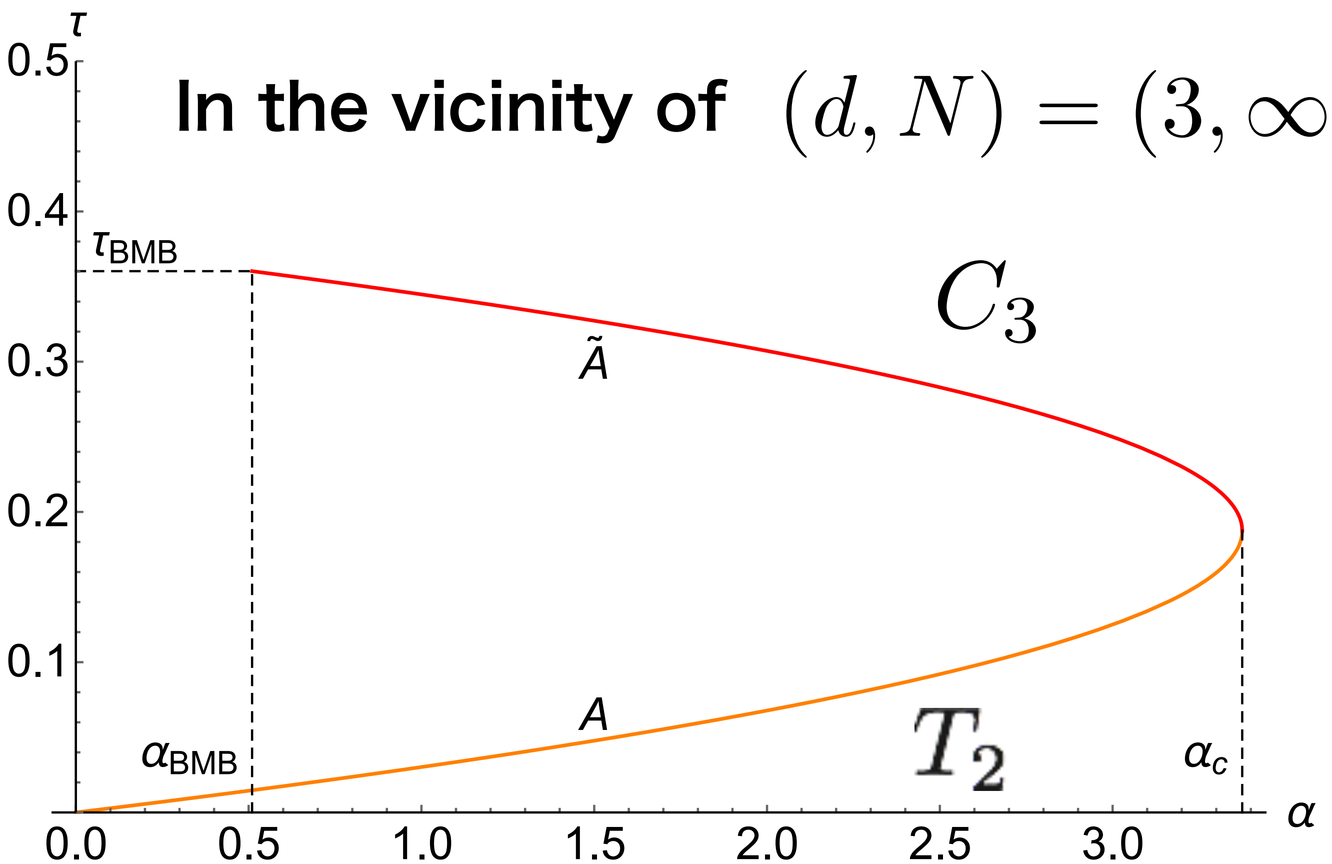
- Derivation: We expand the potential as
$$\bar{V}_{\alpha,N}(\bar{\varrho}) = \bar{V}_{\alpha,N=\infty}(\bar{\varrho}) + \bar{V}_{1,\alpha}(\bar{\varrho})/N + O(1/N^2).$$

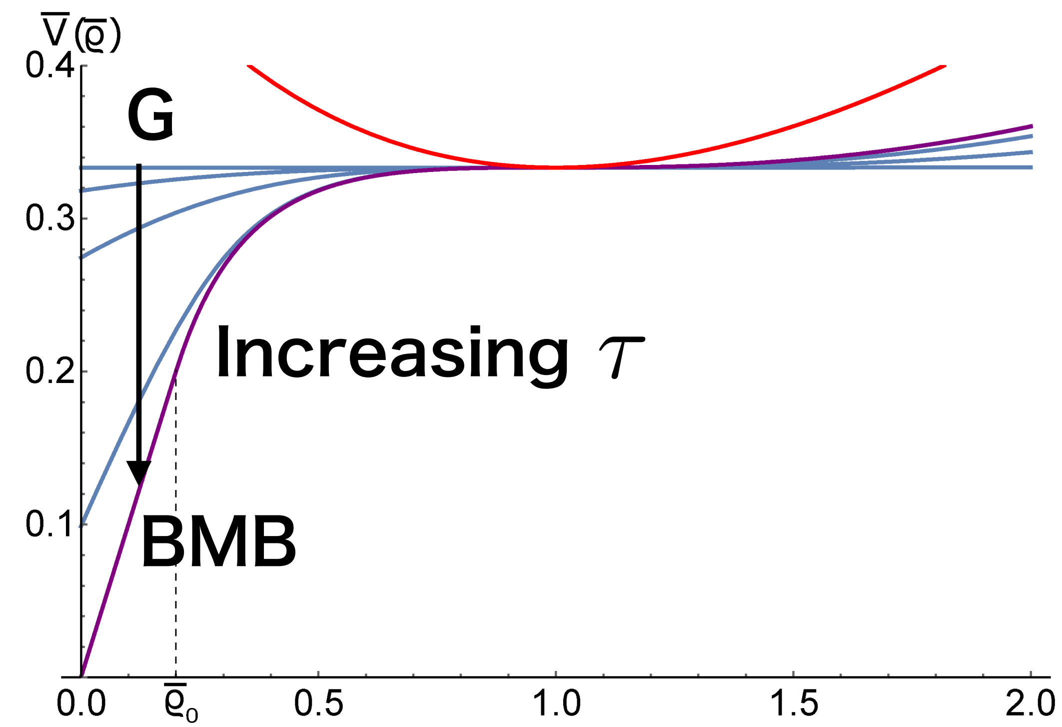
and impose analyticity of $\bar{V}_{1,\alpha}(\bar{\varrho})$ around $\bar{\varrho} = 1$

Plot of τ as a function of α

Under the double limit $d = 3 - \alpha/N, N \to \infty$

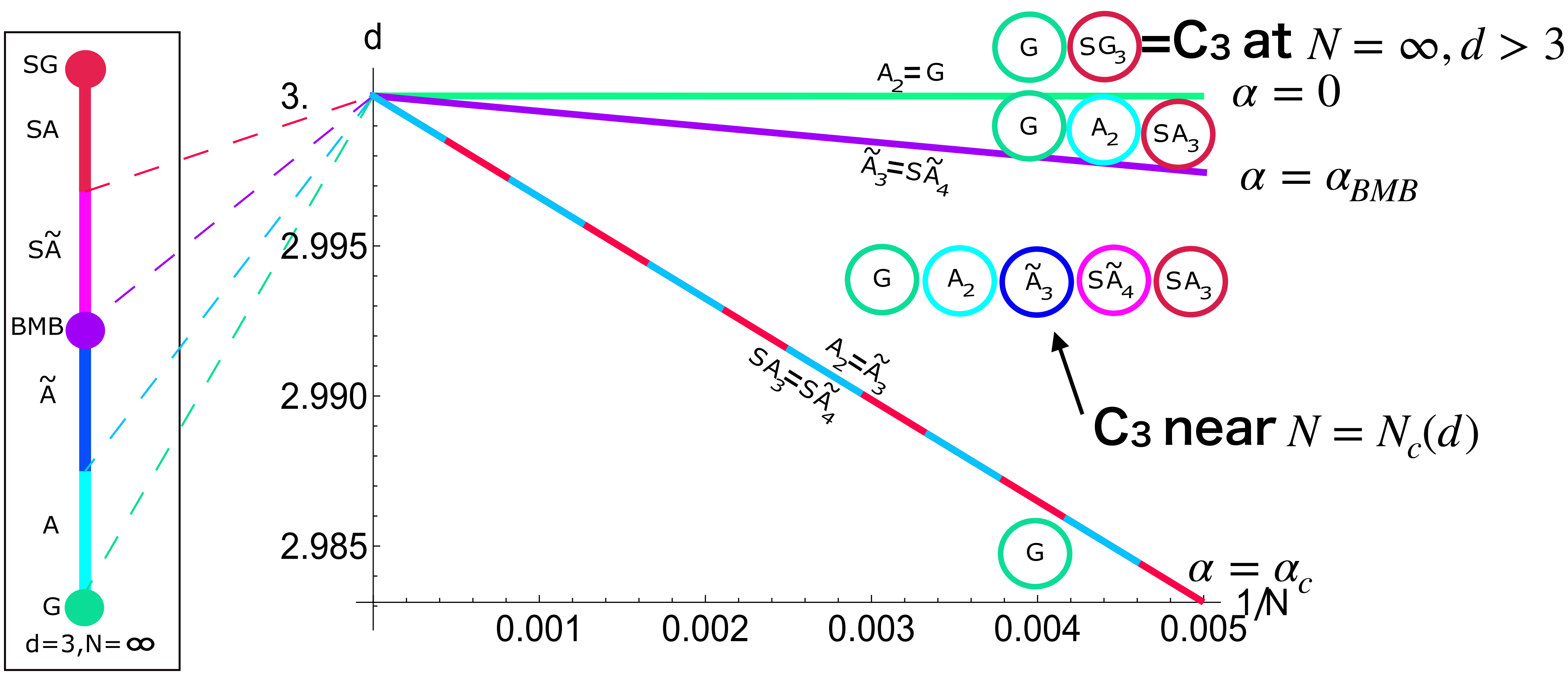
In the vicinity of $(d, N) = (3, \infty)$





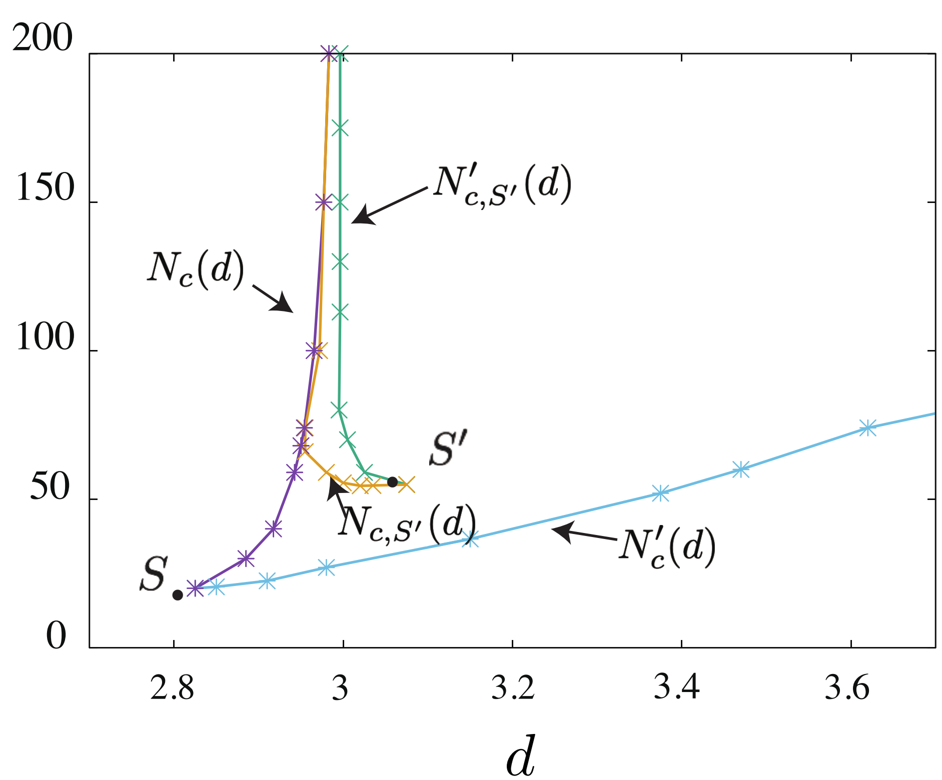
- This relation between is also valid for the FPs on the singular BMB line.

Fixed point structure in the vicinity of $d = 3, N = \infty$



- The number of relevant directions around a FP is indicated with the subscript.

The second double-valued structure



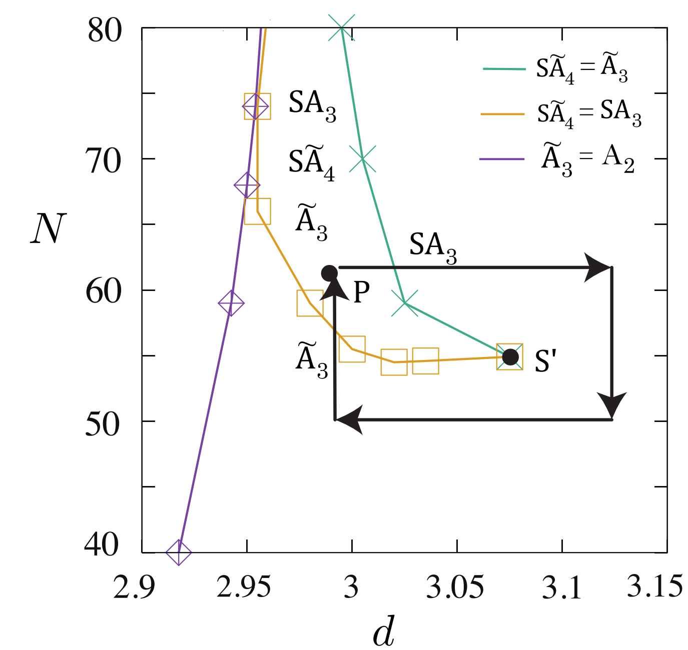


FIG. 11. Point S' and the lines $N_{c,S}(d)[A_2 = \tilde{A}_3]$ (violet diamonds), $N'_{c,S'}(d)[\tilde{A}_3 = S\tilde{A}_4]$(green crosses) and $N_{c,S'}(d)[SA_3 = S\tilde{A}_4]$ (orange squares). Starting from P, SA_3 is followed on a clockwise closed path traveling around S'. When $d > 3$, SA_3 and SG_3 are one and the same FP. SA_3 remains real all along the path but back to the point P, it is $\tilde{A}_3$. Following SA_3 along a path traveling twice around S' it comes back to SA_3.

- This structure is consistent with the points (i)-(iii) and what we found for the BMB line near $d = 3, N = \infty$.

Summary

- We have solved an old paradox about $O(N)$ models at the price of finding a zoo of new FPs and their homotopy structures.
- New multicritical FPs are found in $d = 3$ for $N \gtrsim 28$, whereas the pertubative tricritical FP exists only below $d = 3$.
- We generalized BMB phenomenon for finite-N cases, which turned out to be necessary to have a consistent picture.