Thermal behavior of effective $U_A(1)$ anomaly couplings in reflection of higher topological sectors

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GF and A. Patkos, Phys. Rev. D**105**, 096007 (2022)
GF and A. Patkos, arXiv:2311.02186

Outline

Introduction

- QCD Lagrangian:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{q}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\delta_{ij} \right) q_j$$

$\longrightarrow$ $SU(3)$ gauge symmetry
$\longrightarrow$ $U_L(N_f) \times U_R(N_f)$ global (approx.) chiral symmetry
$\longrightarrow$ anomalous breaking of $U_A(1)$ axial symmetry

- At low temperatures: spontaneous breaking
  $SU_L(N_f) \times SU_R(N_f) \longrightarrow SU_V(N_f)$

Introduction

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- At low temperatures: spontaneous breaking $SU_L(N_f) \times SU_R(N_f) \longrightarrow SU_V(N_f)$
- At high temperatures: $U_A(1)$ anomaly disappears
  $\longrightarrow$ semi-classical approximation of instanton density:
  [ρ - instanton size]

$$n(\rho) \sim \rho^{-5} \exp[-(\pi\rho T)^2]$$

$\longrightarrow$ only valid for $T \gg T_C$
$\longrightarrow$ what is the fate of the anomaly for low temperatures?

3 FLAVOR CHIRAL MESON MODEL:

- Low energy effective model: M - meson fields
  [excitations of M: π, K, η, η' and a_0, κ, f_0, σ]

$$M = (s^a + i\pi^a)T^a \qquad \mathrm{Tr}\,(T^a T^b) = \delta_{ab}/2$$

- Lagrangian with renormalizable operators ($3+1$ dim.): (Euclidean!)

$$\begin{aligned}\mathcal{L} = &\ \mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + m^2\,\mathrm{Tr}\,(M^\dagger M)\\ &+ g_1[\,\mathrm{Tr}\,(M^\dagger M)]^2 + g_2\,\mathrm{Tr}\,(M^\dagger M M^\dagger M)\\ &+ a\,(\,\det M^\dagger + \det M) - \mathrm{Tr}\,[H(M^\dagger + M)]\end{aligned}$$

$\longrightarrow$ $\mathcal{L}$ contains renormalizable operators and it is invariant under chiral symmetry [apart from $U_A(1)$]

$\longrightarrow$ in the quantum effective action every operator allowed by symmetry is present!

Anomalous couplings in the LσM

- Quantum effective action:

$$\Gamma[M] = -\log \int \mathcal{D}\hat{M} \exp\Big(-\int_x \mathcal{L}_J[M+\hat{M}]\Big) - \int_x \mathrm{Tr}\,(J^\dagger M + J M^\dagger)$$

$\longrightarrow$ Ansatz for $\Gamma[M]$? It has to reflect chiral symmetry!

- Chiral invariants for 3 flavors:

$$\begin{aligned} \rho &= \mathrm{Tr}\,(M^\dagger M), \\ \tau &= \mathrm{Tr}\,(M^\dagger M - \rho/3)^2 \\ \rho_3 &= \mathrm{Tr}\,(M^\dagger M - \rho/3)^3 \\ \Delta &= \det M^\dagger + \det M \quad \rightarrow \text{anomaly}\,! \end{aligned}$$

- Note 1: higher order traces are not independent!
- Note 2: $\tilde{\Delta} \equiv \det M^\dagger - \det M$ has wrong parity and $\tilde{\Delta}^2$ is not independent!

Anomalous couplings in the LσM

- What terms besides $\sim \Delta$ can describe the anomaly?

  $\longrightarrow \rho\Delta,\ \rho^2\Delta,\ ...,\ \rho^n\Delta,\ ...$

  $\longrightarrow \tau\Delta,\ \tau^2\Delta,\ ...,\ \tau^n\Delta,\ ...$ $\Rightarrow$ dropped!

  $\longrightarrow \rho_3\Delta,\ \rho_3^2\Delta,\ ...,\ \rho_3^n\Delta,\ ...$ $\Rightarrow$ dropped!

- Chiral limit: in the ground state $M \sim \mathbf{1} \Rightarrow \tau = 0,\ \rho_3 = 0$!

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- Chiral limit: in the ground state $M \sim \mathbf{1} \Rightarrow \tau = 0,\ \rho_3 = 0$!
- Resummation:

$$\sum_n a_n \rho^n \Delta \equiv A(\rho)\Delta$$

$$\sum_n a_n^{(2)} \rho^n \Delta \equiv A^{(2)}(\rho)\Delta^2$$

$$\ldots$$

- Higher orders in $\Delta \Rightarrow$ instantons with $|Q| > 1$
- Resummation in Δ: $U(\rho, \Delta)$

Anomalous couplings in the LσM

- Classical action:

$$S[M] = \int_x \mathcal{L} = \int_x \Big[\mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + m^2\rho + \lambda_1\rho^2 + \lambda_2\tau + a\Delta - \mathrm{Tr}\,[H(M^\dagger + M)] \Big]$$

- Ansatz for the effective action:

$$\Gamma[M] = \int_x \gamma = \int_x \Big[\mathrm{Tr}\,[\partial_i M^\dagger \partial_i M] + U(\rho,\Delta) + C(\rho)\tau - \mathrm{Tr}\,[H(M^\dagger + M)] \Big]$$

- Task: calculate generalized „couplings” $U(\rho,\Delta)$ and $C(\rho)$
- Use FRG with Litim's optimal regulator:
  $R_k(\omega,\vec{q}) = (k^2 - \vec{q}^2)\Theta(k^2 - \vec{q}^2)$

Chirally invariant flow equations

- Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2}\int \tilde{\partial}_k \,\mathrm{Log\ Det}\,(\Gamma_k^{(2)} + R_k)$$

- From the ansatz for Γ_k:

$$\partial_k \Gamma_k|_{\mathrm{hom}} = \int_x \left[\partial_k U_k(\rho,\Delta) + \partial_k C_k(\rho)\tau\right]$$

- Task: match the Wetterich equation with the ansatz

  $\longrightarrow$ problem: $\Gamma_k^{(2)}$ is not chirally invariant (but Γ_k has to be!)
  $\longrightarrow$ any background fields can be applied!

- E.g. $M = (s_0 + i\pi_0)T_0 \sim \mathbf{1} \quad \Rightarrow \quad \tau = 0$

$$\rho = (s_0^2 + \pi_0^2)/2$$
$$\Delta = s_0(s_0^2 - 3\pi_0^2)/(3\sqrt{6})$$

- Multiplet structure: $8 \oplus 8 \oplus \{1 \text{ doublet}\}$

Chirally invariant flow equations

$$\partial_k U_k(\rho,\Delta) = \Omega_d \frac{4k^d}{d} T \sum_n \tilde{\partial}_k \log\left((\omega_n^2 + k^2 + U_{k,\rho})^2 + \frac{4}{3}(\omega_n^2 + k^2 + U_{k,\rho})\rho C_k - \frac{1}{3}\rho U_{k,\Delta}^2 + 2\Delta C_k U_{k,\Delta} \right)$$
$$+ \Omega_d \frac{k^d}{2d} T \sum_n \tilde{\partial}_k \log\Big[(\omega_n^2 + k^2 + U_{k,\rho} + 3\Delta U_{k,\rho\Delta})^2 + (\omega_n^2 + k^2 + U_{k,\rho})\left(2\rho U_{k,\rho\rho} + \frac{2}{3}\rho^2 U_{k,\Delta\Delta}\right)$$
$$-6\Delta U_{k,\Delta} U_{k,\rho\rho} - \frac{4}{3}\rho U_{k,\Delta}(U_{k,\Delta} + 2\rho U_{k,\rho\Delta}) - \frac{4}{3}\rho^3 U_{k,\rho\Delta}^2$$
$$-U_{k,\Delta\Delta}\left(2\rho\Delta U_{k,\Delta} - \frac{1}{3}(4\rho^3 - 27\Delta^2)U_{k,\rho\rho}\right)\Big],$$

- By subtracting the flow of U_k and choosing a different background, one identifies $\tau \longrightarrow$ yields the flow of C_k
- E.g. $M = i(\pi_0 T_0 + \pi_8 T_8) \quad \Rightarrow \quad \tau = \pi_0^2\pi_8^2/3 + \mathcal{O}(\pi_8^3)$

  $\rho = (\pi_0^2 + \pi_8^2)/2$

  $\Delta = 0$

  $\longrightarrow$ more complicated multiplet structure, but the Log Det combines into invariants

Chirally invariant flow equations

$$
\begin{aligned}
\partial_k C_k(\rho) \;=\; & \Omega_d \frac{k^d}{d} T \sum_n \tilde{\partial}_k \Bigg\{ \frac{U_{k,\Delta}^2}{2\rho}\left(\frac{1}{D_0}-\frac{1}{2D_8}-\frac{(\omega_n^2+k^2+U_{k,\rho})^2}{2D_0D_8}\right) \\
& +\frac{3}{4\rho}(\omega_n^2+k^2+U_{k,\rho})\left(U_{k,\rho\rho}-\frac{2}{3}C_k\right)\left(\frac{1}{D_8}-\frac{1}{D_0}\right) \\
& +\frac{1}{2D_0}\left[\rho U_{k,\Delta\Delta}\left(\frac{2}{3}C_k-U_{k,\rho\rho}\right)+U_{k,\rho\Delta}(2U_{k,\Delta}+\rho U_{k,\rho\Delta})\right] \\
& -\frac{1}{2D_0D_8}\Bigg[(\omega_n^2+k^2+U_{k,\rho})\left(2U_{k,\rho\rho}\left(\frac{1}{2}U_{k,\Delta}-\rho U_{k,\rho\Delta}\right)^2+\frac{2}{3}(2C_k+\rho U_{k,\Delta\Delta})\left(\frac{1}{2}U_{k,\Delta}+\rho U_{k,\rho\Delta}\right)^2\right) \\
& +2(\omega_n^2+k^2+U_{k,\rho})^2\rho U_{k,\rho\Delta}^2+\frac{2}{9}\rho^2 U_{k,\Delta\Delta}C(U_{k,\Delta}+2\rho U_{k,\rho\Delta})^2\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& -\Omega_d\frac{k^d}{2d}T\sum_n\tilde{\partial}_k\Bigg\{\frac{1}{3D_0D_8}\Bigg[(\omega_n^2+k^2+U_{k,\rho}+2\rho U_{k,\rho\rho})\left(\omega_n^2+k^2+U_{k,\rho}+\frac{4}{3}\rho C\right)(2C_k-\rho U_{k,\Delta\Delta})^2 \\
& +\left(\omega_n^2+k^2+U_{k,\rho}+\frac{2}{3}\rho^2U_{k,\Delta\Delta}\right)(\omega_n^2+k^2+U_{k,\rho})(3U_{k,\rho\rho}+4(C_k+\rho C_{k,\rho}))^2\Bigg] \\
& -\frac{1}{3D_0D_8}(3U_{k,\rho\rho}+4(C_k+\rho C_{k,\rho}))\Big[U_{k,\Delta}\left(\omega_n^2+k^2+U_{k,\rho}+\frac{2}{3}\rho^2U_{k,\Delta\Delta}\right)(2\rho U_{k,\rho\Delta}+U_{k,\Delta}) \\
& +2(U_{k,\Delta}+\rho U_{k,\rho\Delta})(\omega_n^2+k^2+U_{k,\rho})(2\rho U_{k,\rho\Delta}-U_{k,\Delta})\Big] \\
& +\frac{1}{3D_0D_8}(2C_k-\rho U_{k,\Delta\Delta})\Big[(\omega_n^2+k^2+U_{k,\rho}+2\rho U_{k,\rho\rho})U_{k,\Delta}(2\rho U_{k,\rho\Delta}-U_{k,\Delta}) \\
& +2\left(\omega_n^2+k^2+U_{k,\rho}+\frac{4}{3}\rho C\right)(U_{k,\Delta}+\rho U_{k,\rho\Delta})(2\rho U_{k,\rho\Delta}+U_{k,\Delta})\Big] \\
& -\frac{4}{3D_0D_8}\left[\frac{1}{4}U_{k,\Delta}^2-\rho^2U_{k,\rho\Delta}^2+\rho(2C_k-\rho U_{k,\Delta\Delta})\left(U_{k,\rho\rho}+\frac{4}{3}(C_k+\rho C_{k,\rho})\right)\right]U_{k,\Delta}(U_{k,\Delta}+\rho U_{k,\rho\Delta})\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& +\Omega_d\frac{k^d}{d}T\sum_n\Bigg\{\Big[\frac{7}{D_8}\left(C_k^2+(\omega_n^2+k^2+U_{k,\rho})C_{k,\rho}+\frac{2}{3}\rho C_kC_{k,\rho}\right) \\
& -\frac{8}{3D_8^2}\left(\frac{1}{2}U_{k,\Delta}^2+\frac{2}{3}\rho C_k^2+2C(\omega_n^2+k^2+U_{k,\rho})\right)^2\Big] \\
& +\frac{1}{2D_0}\Bigg[\left(C_{k,\rho}-\frac{3}{2}U_{k,\Delta\Delta}\right)(\omega_n^2+k^2+U_{k,\rho}+2\rho U_{k,\rho\rho}) \\
& +(5C_{k,\rho}+2\rho C_{k,\rho\rho})\left(\omega_n^2+k^2+U_{k,\rho}+\frac{2}{3}\rho^2U_{k,\Delta\Delta}\right)+3U_{k,\rho\Delta}(U_{k,\Delta}+\rho U_{k,\rho\Delta})\Bigg] \\
& +\frac{1}{2D_8}\Bigg[(\omega_n^2+k^2+U_{k,\rho})\left(6C_{k,\rho}+\frac{1}{2}U_{k,\Delta\Delta}\right)-U_{k,\Delta}U_{k,\rho\Delta}+2C_k^2+\frac{4}{3}\rho C_k\left(C_{k,\rho}+\frac{1}{2}U_{k,\Delta\Delta}\right) \\
& -\frac{1}{3D_8}\left[4C_k(\omega_n^2+k^2+U_{k,\rho})+U_{k,\Delta}^2+\frac{4}{3}\rho C_k^2\right]^2\Bigg]\Bigg\},
\end{aligned}
$$

Chirally invariant flow equations

- Numerical solution: grid method
- Start from $k = \Lambda \equiv 1$ GeV and integrate toward $k \longrightarrow 0$ ($k_{\rm end} \sim 50\,\mathrm{MeV}$)
- 2D grid is costly $\Rightarrow U(\rho, \Delta) \approx U(\rho) + A(\rho)\Delta + B(\rho)\Delta^2$

Chirally invariant flow equations

- Numerical solution: grid method
- Start from $k = \Lambda \equiv 1$ GeV and integrate toward $k \longrightarrow 0$ ($k_{\mathrm{end}} \sim 50\,\mathrm{MeV}$)
- 2D grid is costly $\Rightarrow U(\rho, \Delta) \approx U(\rho) + A(\rho)\Delta + B(\rho)\Delta^2$
- Classical (UV) potential:

$$V_\Lambda = m^2\rho + \lambda_1\rho^2 + \lambda_2\tau + a\Delta - h_a s^a$$

- Effective potential:

$$V_k = U_k(\rho) + C_k(\rho)\tau + A_k(\rho)\Delta + B_k(\rho)\Delta^2 - h_a s^a$$

- Six unknown parameters $\{m^2, g_1, g_2, a, h_{\mathrm{s}}, h_{\mathrm{ns}}\} \Rightarrow$ six inputs!
- 2 PCAC relations (f_π, f_K decay const.'s), 4 masses (π, K, η, η')

Chirally invariant flow equations

- Effective model valid for scales $0 \lesssim k \lesssim 1\,\mathrm{GeV}$
- UV model parameters $\{m^2, g_1, g_2, a, h_\mathrm{s}, h_\mathrm{ns}\}$ should (in principal) be temperature dependent
  $\longrightarrow$ usually ignored
- BUT: T dependence of the anomaly is crucial![1]

$$a(T) \sim \int d\rho \rho^{3N_f} n(\rho) \sim \int d\rho \rho^4 \exp[-(\pi \rho T)^2] d\rho \sim T^{-5}$$

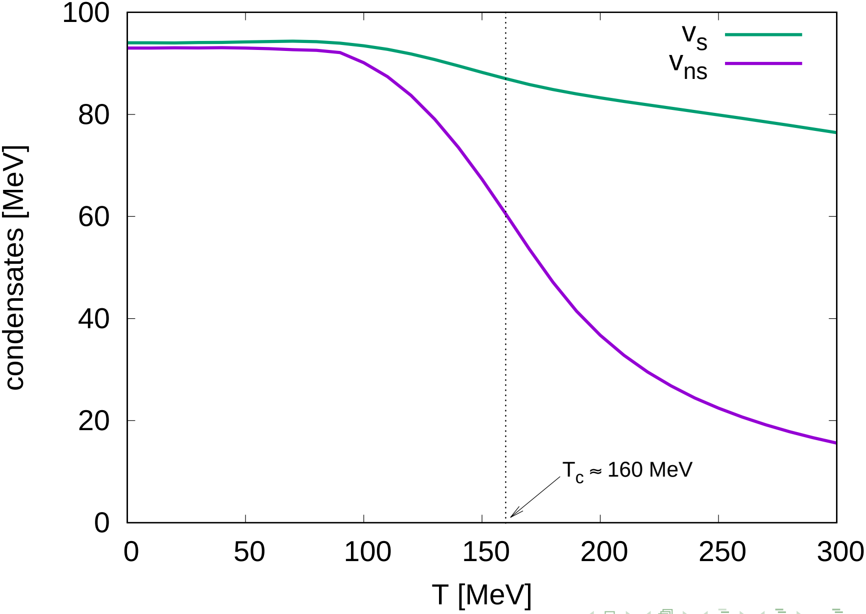
- Working hypothesis:

$$a(T) = a(T=0)\left[1 + \left(\left(\frac{T_c}{T}\right)^5 - 1\right)\Theta(T - T_c)\right]$$

$\longrightarrow$ not smooth at T_c
$\Rightarrow$ non-physical break points are expected

[1] R. D. Pisarski & F. Rennecke, Phys. Rev. D**101**, 114019 (2020).

Numerical results



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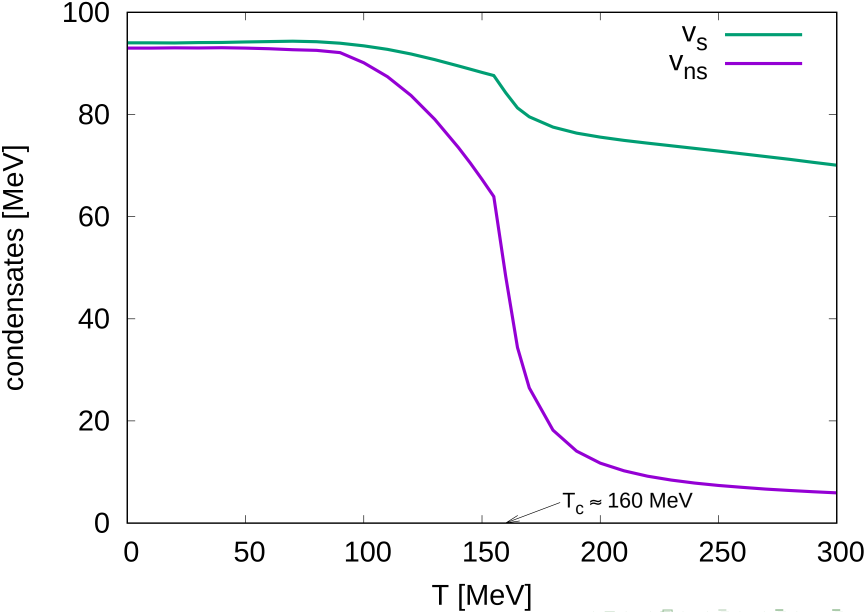
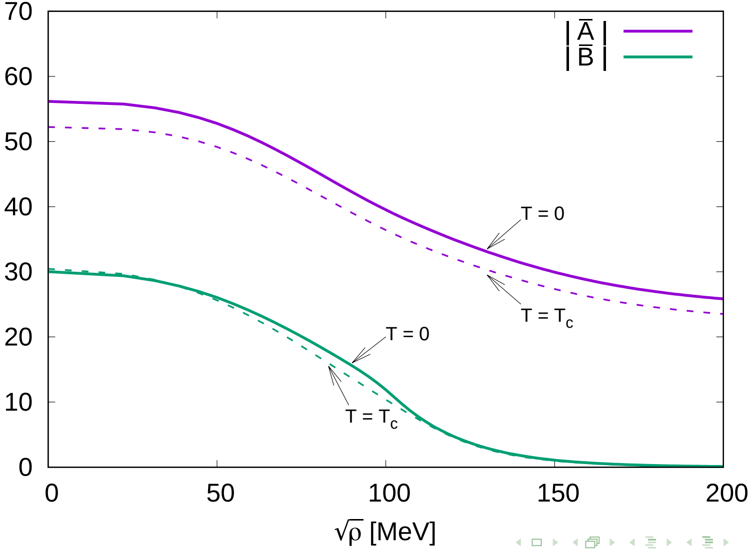
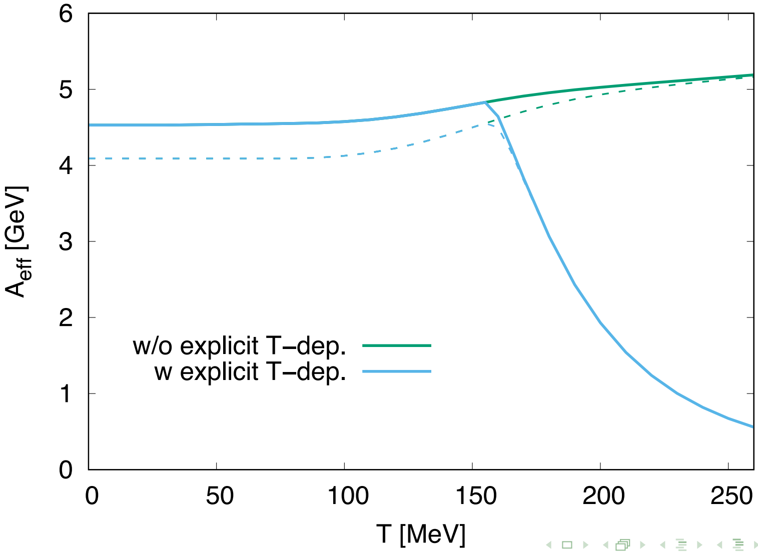


Numerical results

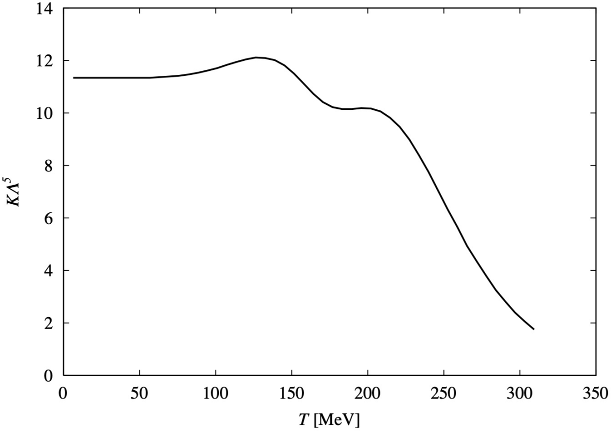
- Rescaled anomaly functions: $\bar{A} = A/k$, $\bar{B} = Bk^2$



- Effective anomaly coupling: $A_{\text{min}}\Delta_{\text{min}} + B_{\text{min}}\Delta^2_{\text{min}} \equiv A_{\text{eff}}\Delta_{\text{min}}$



- Similar behavior in *K. Fukushima et. al*, PRC**63** (2001) 045203
  $\longrightarrow$ NJL model calculation: fitting the *K* anomaly parameter to recover $\chi_{\rm top}$ of the lattice



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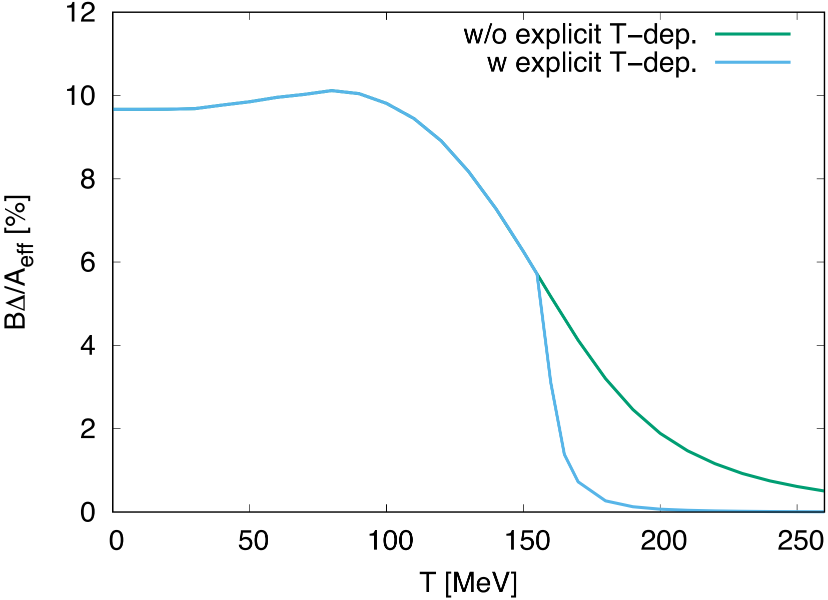
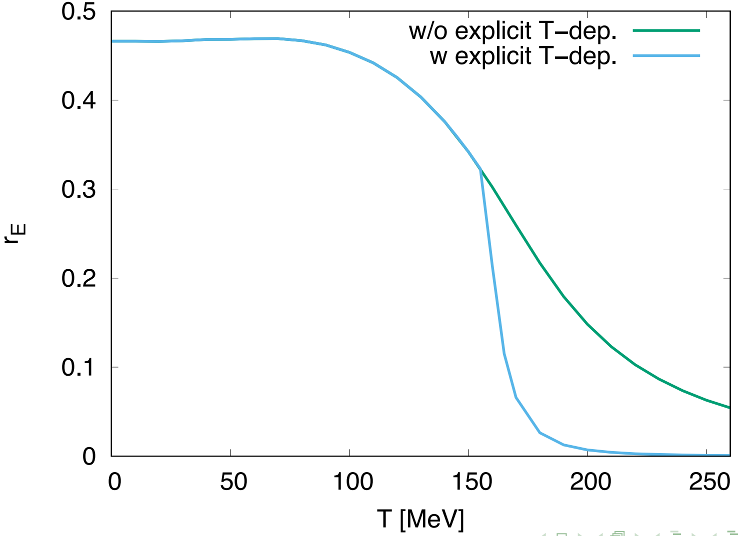


- r_E ratio: $r_E = |A(\rho)\Delta + B(\rho)\Delta^2| / (U(\rho) + C(\rho)\tau)\Big|_{\rm min}$



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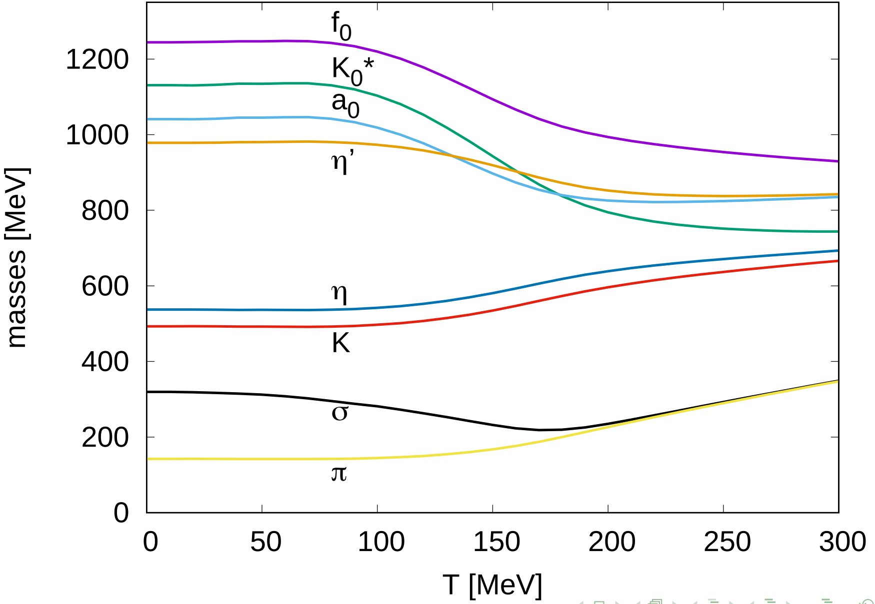

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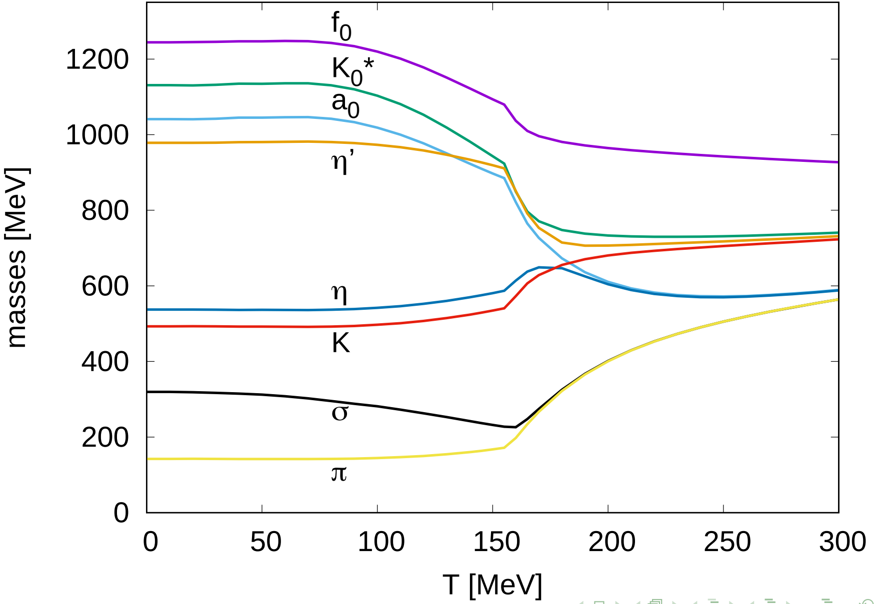

Summary

- Thermal evolution of the $U_A(1)$ anomaly in LσM ($N_f = 3$)
- Infinite class of $U_A(1)$ breaking operators
  $\longrightarrow$ inclusion of topological sectors with arbitrary charge
- Resulting anomaly function calculated via the FRG
  $\longrightarrow$ exploring the condensate dependence of the $|Q| = 1$ and $|Q| = 2$ couplings
- Mesonic fluctuations strengthen the anomaly toward T_C
  $\longrightarrow$ temperature dependence of the bare anomaly parameter is important!
- $\sim 10\%$ increase of the effective anomaly coupling toward T_C
- a similar ratio of $\sim 10\%$ is coming from the $|Q| = 2$ coupling

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- Future plans:
  $\longrightarrow$ calculate the full $U(\rho, \Delta)$ function on a 2D grid
  $\longrightarrow$ effective model(s) for $N_c = 2$ QCD (Pauli-Gursey symm.)