# Gradient Flow Exact Renormalization Group for Scalar Quantum Electrodynamics

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# Summary

- Gradient Flow Exact Renormalization Group defines the Wilsonian effective action based on diffusion equations
- (Massless) Scalar Quantum Electrodynamics with off-shell BRST invariance is studied
- Modified BRST invariance reduces to the ordinary Ward-Takahashi identity if ghost sector is free
- RG flow equation is perturbatively solved up to second order of electric charge
- Consistent results with the perturbation theory
- GF-ERG can give a gauge-invariant RG flow

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# Exact Renormalization Group

- Framework to study physics under varying energy scale
- Wilsonian effective action describes physics at the energy scale $\Lambda := \Lambda_0 e^{-\tau}$

  ($\Lambda_0$: UV cutoff)
- $S_\tau$ is intuitively defined by integrating out higher momentum mode:

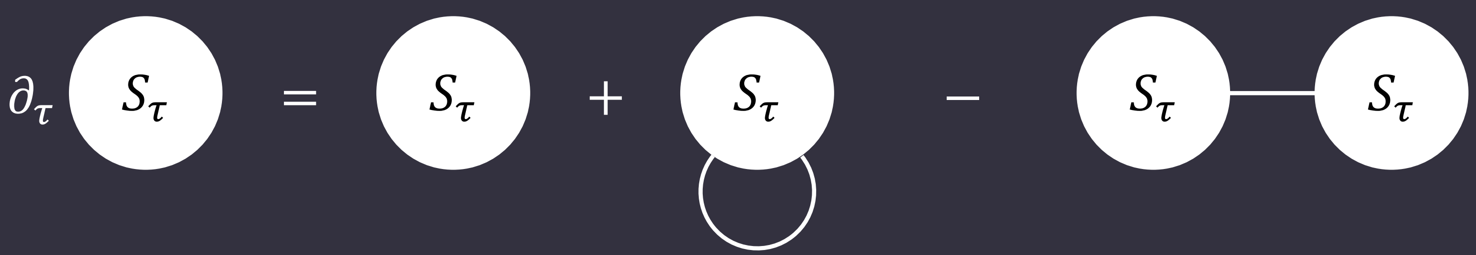
$$e^{-S_\tau} := \int D\phi_{p>\Lambda} e^{-S_0}$$

- $\tau$-dependence of $S_\tau$ is described by a functional differential eq. → "ERG equation"

# Wilson-Polchinski equation

- Typical example of ERG equation:

$$\partial_\tau S_\tau = \int_p \left\{ \left[ \left(2p^2 + \frac{D+2-2\gamma_\tau}{2}\right) + p_\mu \frac{\partial}{\partial p_\mu} \right] \phi(p) \frac{\delta S_\tau}{\delta\phi(p)} + (2p^2+1-\gamma_\tau)\left( \frac{\delta^2 S_\tau}{\delta\phi(p)\delta\phi(-p)} - \frac{\delta S_\tau}{\delta\phi(p)}\frac{\delta S_\tau}{\delta\phi(-p)}\right)\right\}$$




- ERG equation nonperturbatively defines an RG flow

($\gamma_\tau$: anomalous dimension) [J.Polchinski Nucl.Phys.B 231 (1984) 269-295]

(We use dimensionless notations on D-dimensional Euclidean spacetime)

# Gauge Invariance in ERG

- Wilsonian effective action with a naive UV cutoff is NOT consistent with gauge invariance

- Gauge transformation mixes higher and lower momentum modes

$$A_\mu^a(p) \to A_\mu^a(p) - p_\mu \omega^a(p) - i f^{abc} \int_q \omega^b(p-q) A^c(q)$$

- Can we define the Wilsonian effective action in a manifestly gauge-invariant way?

# Renormalization and Diffusion

- Solution to the WP equation

$$e^{S_\tau[\phi]} = \hat{s}_\phi^{-1} \int D\phi' \prod_{x,i} \delta(\phi(x) - e^{\tau(D-2)/2} Z_\tau^{1/2} \varphi'(t, xe^\tau))\, \hat{s}_{\phi'} e^{S_0[\phi']}$$

$\left(\hat{s}_\phi \coloneqq \exp\left(-\frac{1}{2}\int_x \frac{\delta^2}{\delta\phi^2}\right)\right.$: "scrambler")

- $\varphi'$ is the solution to the simple diffusion eq.:

$$\partial_t \varphi'(t,x) = \partial_x^2 \varphi'(t,x),\ \ \varphi'(0,x) = \phi'(x)$$

where $t \coloneqq e^{2\tau} - 1$

- Coarse-graining along the diffusion equation can define an RG flow?

# Gradient Flow (GF)

- This one-parameter deformation of fields via the diffusion equation has been studied in the context of "gradient flow"

- The gradient flow is a method to construct composite operators without the equal-point singularity

- Correlation functions are UV finite with wave function renormalization:

$$Z_t^{-n/2}\langle\varphi(t,x_1)\varphi(t,x_2)\cdots\varphi(t,x_n)\rangle_\phi < \infty$$

even for the equal point case (e.g. $x_1 = x_2$)

[F.Capponi,L.Debbio,S.Ehret,R.Pellegrini,A.Rago 1512.02851]

# GF for gauge fields

- Gradient flow equation for gauge fields

[M.Lüscher,P.Weisz 1101.0963]

$$\partial_t B'_\mu = D'_\nu G'_{\nu\mu} + \alpha_0 D'_\mu \partial_\nu B'_\nu$$

with $B'_\mu(0,x) = A'_\mu(x)$

($\alpha_0$: real number)

- Correlation functions of $B'_\mu$ are UV finite without any additional renormalization
- $D'_\nu G'_{\nu\mu}$ is expanded as

$$D'_\nu G'_{\nu\mu} = \partial_x^2 B'_\mu + \cdots$$

$\Rightarrow$ the above GF equation is regarded as a "gauge-covariant" diffusion equation

# Gradient Flow-ERG (GF-ERG)

- Framework to define an RG flow based on coarse-graining along diffusion equations

[H. Sonoda and H. Suzuki 2012.03568]

$$e^{-S_\tau[A]} := \hat{s}_A^{-1} \int DA' \prod_{x',a,\mu} \delta\left(A_\mu^a(x) - e^{\int_\tau (D-2+2\gamma_\tau)/2} B_\mu'^a(t, x' e^\tau)\right) \hat{s}_{A'} e^{-S_0[A']}$$

$\left(\hat{s}_A := \exp\left[-\frac{1}{2}\int_x \frac{\delta^2}{\delta A_\mu^a \delta A_\mu^a}\right]\right.$: scrambler operator)

- Symmetries of the diffusion eq. are inherited by the RG flow
  $\Rightarrow$ $S_\tau$ is gauge-invariant at an arbitrary energy scale!

# GF-ERG equation

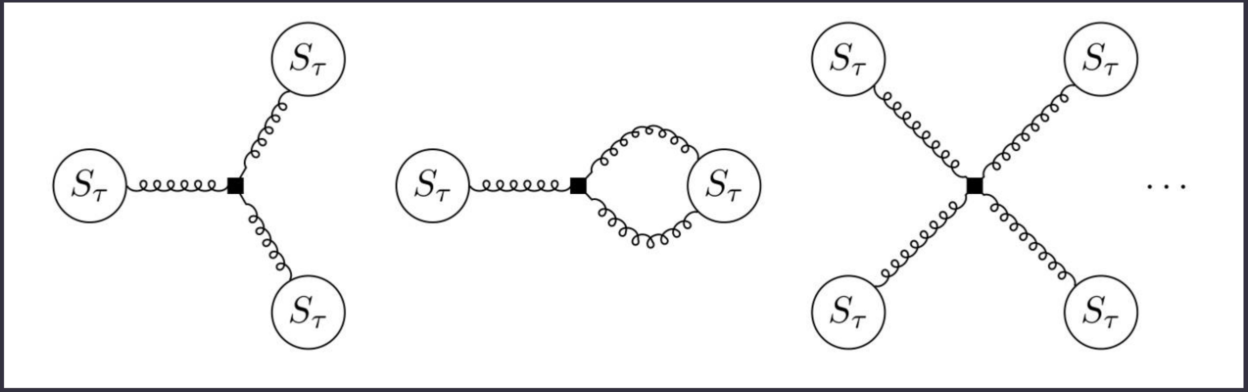
- The counterpart in GF-ERG to the ERG equation:

$$\partial_\tau e^{-S_\tau} =$$

$$\int_x \frac{\delta}{\delta A_\mu^a}\left[-2D_\nu F_{\nu\mu}^a - 2\alpha_0 D_\mu \partial_\nu A_\nu^a - \left(\frac{D-2+2\gamma_\tau}{2} + x^\mu \frac{\partial}{\partial x^\mu}\right) A_\mu^a\right]_{A\to A+\delta/\delta A} e^{-S_\tau}$$

which is called "GF-ERG equation"

- Third/fourth-order functional derivative is included
  $\Rightarrow$ three/four-point vertices arise in diagrams
  (c.f. up to second-order derivatives in WP eq.)



# Recent Studies

- Inclusion of fermions in QED
  Chiral symmetry, Axial anomaly [Y.Miyakawa and H.Suzuki 2106.11142]
  [Y.Miyakawa 2201.08181]

- Gauge fixing, ghost, perturbative analysis around Gaussian fixed point in QED
  [Y.Miyakawa, H.Sonoda and H.Suzuki 2111.15529]

- GF-ERG eq. and gauge symmetry of 1PI effective action
  [H.Sonoda and H.Suzuki 2201.04448]

- RG invariance of chiral anomaly as a composite operator [Y.Miyakawa, H.Sonoda and H.Suzuki 2304.14753]

- Fixed point structure of scalar field theories
  [Y.Abe, Y.Hamada and JH 2201.04111]

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# Our Motivation

- Perturbative analysis of QED
  - On-shell BRST invariance
  - Loop corrections to photon kinetic term are consistent with perturbation theory in four dims.

[Y.Miyakawa, H.Sonoda and H.Suzuki 2111.15529]

- Question
  - Should Off-shell BRST invariance be respected? (Wilsonian effective action itself obtains quantum corrections)
  - Photon mass vanishes in general dimensions? (non-zero in the conventional ERG case)

- Scalar QED (sQED) has more simple structure
  ⇒ Suitable to study these points

# Our Work

- We derive GF-ERG flow of massless sQED with diffusion eqs. consistent to off-shell BRST transformation
- Modified BRST invariance is derived
  $\Rightarrow$ The ordinary Ward-Takahashi identity if the ghost sector is free
- GF-ERG eq. is solved perturbatively up to second-order in electric charge
- Beta function (anomalous dimension) is consistent with the perturbation theory in four dimensions
- One-loop correction to photon mass vanishes in general dimensions

# Diffusion Equation

- Diffusion eqs. consistent with the off-shell BRST trf.

$$\left.\begin{aligned}\partial_t A_\mu &= \partial_x^2 A_\mu \\ \partial_t c &= \partial_x^2 c \\ \partial_t \bar{c} &= \partial_x^2 \bar{c} \\ \partial_t B &= \partial_x^2 B\end{aligned}\right\} \text{same as WP eq.}$$

$$\left.\partial_t \phi = D_\mu D_\mu \phi + ie_0(\partial_\mu A_\mu)\phi\right\} \text{differs from WP eq.}$$

- The flow along these equations are commutative with the BRST transformation (i.e. $\partial_t(\delta_B A_\mu) = \delta_B(\partial_t A_\mu)$ etc.)

$$\left(\begin{aligned}\delta_B A_\mu &= \partial_\mu c \\ \delta_B c &= 0 \\ \delta_B \bar{c} &= B \\ \delta_B B &= 0 \\ \delta_B \phi &= ie_0 c\phi\end{aligned}\right)$$

$\Rightarrow$ Inherited by the RG flow

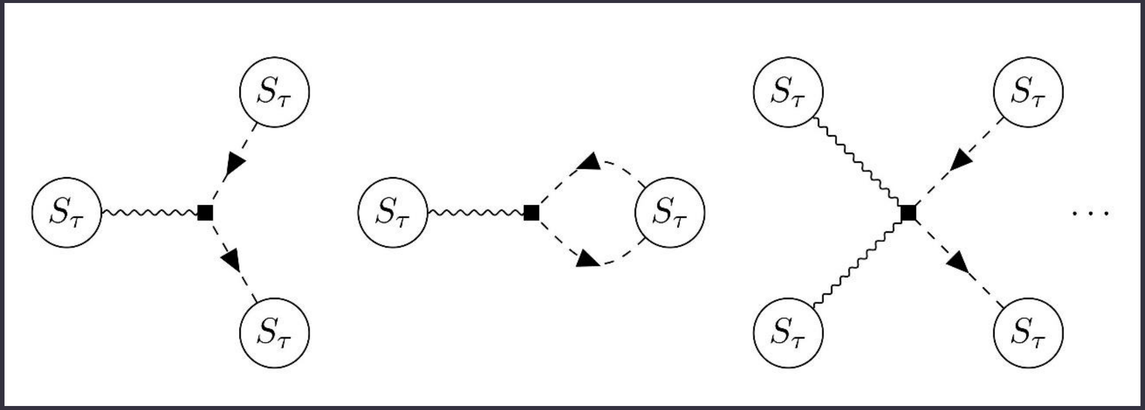
# GF-ERG equation

$$\partial_\tau e^{-S_\tau} = (\text{WP part})$$
$$+ \int_x \frac{\delta}{\delta\phi^*}\left[-4ie_\tau\left(A_\mu + \frac{\delta}{\delta A_\mu}\right)\partial_\mu + 2e_\tau^2\left(A_\mu + \frac{\delta}{\delta A_\mu}\right)^2\right]\left(\phi^* + \frac{\delta}{\delta\phi}\right)e^{-S_\tau} + (\text{c.c.})$$

- Electric charge $e_\tau$ is defined as

$$e_\tau := e_0 \exp\left(-\int_{\tau_0}^{\tau} d\tau'(D - 4 + 2\gamma_{\tau'})/2\right)$$

- GF-ERG eq. includes third-/forth-order functional derivatives



# Modified BRST Invariance

- GF-ERG eq. is invariant under modified BRST trf. due to the scrambler operator:

$$0 = \widetilde{\hat{\delta}_B} e^{-S_\tau} = \hat{s}^{-1}\hat{\delta}_B \hat{s} e^{-S_\tau}$$

$$= \int_x \left[\partial_\mu \left(c + \frac{\delta}{\delta \bar{c}}\right) \cdot \frac{\delta}{\delta A_\mu} + \left(B - \frac{\delta}{\delta B}\right)\frac{\delta}{\delta \bar{c}} + ie_\tau \left(c + \frac{\delta}{\delta \bar{c}}\right)\left(\phi \frac{\delta}{\delta \phi} - \phi^* \frac{\delta}{\delta \phi^*}\right)\right] e^{-S_\tau}$$

- Result I: Assuming the ghost sector is free

$$S_\tau = S_0 + S_I[\mathcal{A}_\mu, \mathcal{B}, \phi, \phi^*],$$

it reduces to the ordinary WT identity with $e^{-k^2}\mathcal{A}_\mu(k)$

$$0 = e^{k^2} k_\mu \frac{\delta S_I}{\delta \mathcal{A}_\mu(k)} + ie_\tau \int_p \left[\phi(p+k)\frac{\delta S_I}{\delta \phi(p)} - \phi^*(p)\frac{\delta S_I}{\delta \phi^*(p+k)}\right]$$

$$\mathcal{A}_\mu(k) \coloneqq e^{k^2}\left(A_\mu(k) + \frac{\delta S_0}{\delta A_\mu(-k)}\right), \mathcal{B}\,(k) \coloneqq e^{k^2}\left(B(k) + \frac{\delta S_0}{\delta B(-k)}\right)$$

$S_0$: Gaussian fixed-point action

# Gaussian Fixed Point

- Fixed point condition with $e_\tau = 0$

$$0 = \int_p \left[ \left( 2k^2 + \frac{D+2}{2} \right) + k \cdot \partial_k \right] A_\mu \cdot \frac{\delta S_0}{\delta A_\mu} - (2k^2 + 1) \frac{\delta S_0}{\delta A_\mu} \frac{\delta S_0}{\delta A_\mu}$$

$$+ \left( A_\mu \to c, \bar{c}, B, \phi, \phi^* \right)$$

- Gaussian fixed-point action

$$S_0 = \frac{1}{2} \int_k A_\mu(k) A_\nu(-k) \left( \frac{k^2 \delta_{\mu\nu} - k_\mu k_\nu}{k^2 + e^{-2k^2}} + \frac{k_\mu k_\nu}{k^2 + \xi e^{-2k^2} + e^{-4k^2}} \right)$$

$$+ (c, \bar{c}, B, \phi, \phi^* \text{ part})$$

- $S_0$ satisfies the modified BRST invariance
- $R_\xi$ gauge with the gauge parameter $\xi$

# Perturbative Analysis

- Set $S_\tau = S_0 + S_I$ and expand $S_I$ with respect to $e_\tau$:

$$S_I = e_\tau S_1 + e_\tau^2 S_2 + \cdots$$

- Assume $S_i$ has no explicit $\tau$-dependence:

$$\partial_\tau S_I = \partial_\tau e_\tau \cdot \frac{\partial S_I}{\partial e_\tau} = -\left(\frac{D-4}{2} + \gamma_\tau\right) e_\tau \frac{\partial S_I}{\partial e_\tau}$$

- To do
  1. Substitute the ansatz into GF-ERG eq. and focus on the terms of each order of $e_\tau$
  2. Solve the RG equation order by order
  3. Determine integral constants by WT identity

# Dressed Fields $(\mathcal{A}_\mu, \Phi)$

- Fundamental variables in perturbative analysis ("fields equipped with external lines")

$$\mathcal{A}_\mu(k) \coloneqq e^{k^2}\left(A_\mu(k) + \frac{\delta S_0}{\delta A_\mu(-k)}\right) = e^{-k^2}\left(h_{\mu\nu}(k)A_\nu(k) + h_B(k^2)ik_\mu B(k)\right)$$

$$\Phi(k) \coloneqq e^{k^2}\left(\phi^*(k) + \frac{\delta S_0}{\delta \phi(k)}\right) = e^{-k^2}h_S(k)\phi(k)$$

$$\left(\begin{aligned} h_{\mu\nu}(k) &\coloneqq \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2 + e^{-2k^2}} + \frac{k_\mu k_\nu}{k^2}\frac{\xi + e^{-2k^2}}{k^2 + \xi e^{-2k^2} + e^{-4k^2}} \\ h_B(k^2) &\coloneqq \frac{1}{k^2 + \xi e^{-2k^2} + e^{-4k^2}} \\ h_S(k^2) &\coloneqq \frac{1}{k^2 + e^{-2k^2}} \end{aligned}\right)$$

# First order

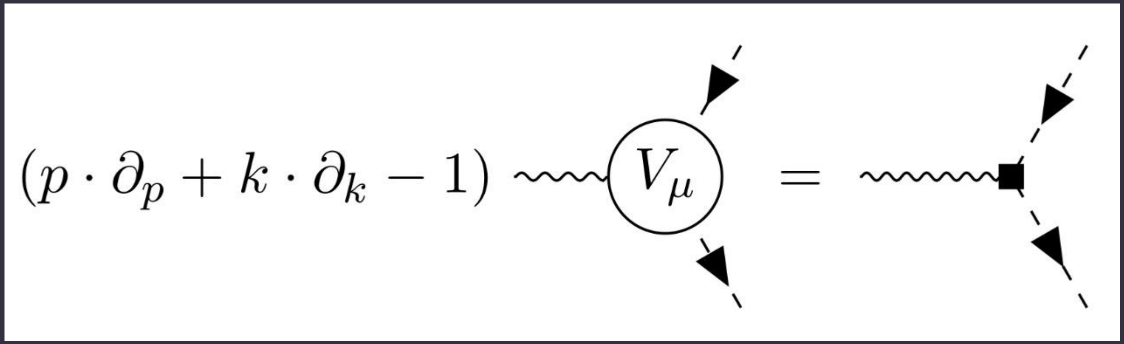
- Ansatz

$$S_1 = \int_{p,k} V_\mu(p,k)\Phi^*(p+k)\mathcal{A}_\mu(p)\Phi(k)$$

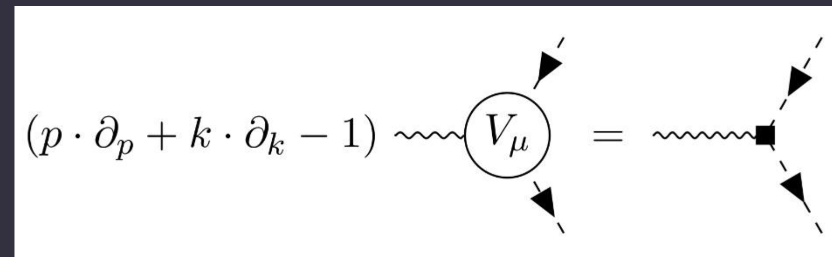
- GF-ERG equation

$$\int_k \left[\left(k\cdot\partial_k + \frac{D+2}{2}\right)\mathcal{A}_\mu \cdot \frac{\delta S_1}{\delta \mathcal{A}_\mu} + \left(k\cdot\partial_k + \frac{D+2}{2}\right)\Phi^* \cdot \frac{\delta S_1}{\delta \Phi^*} + \left(k\cdot\partial_k + \frac{D+2}{2}\right)\Phi \cdot \frac{\delta S_1}{\delta \Phi}\right]$$

$$= -4\int_{p,k} \left(e^{p^2-(p+k)^2-k^2} p^2 (p+k)_\mu + e^{(p+k)^2-p^2-k^2}(p+k)^2 p_\mu\right)$$



# Three-point function




- RG equation

$$(p \cdot \partial_p + k \cdot \partial_k - 1) V_\mu(p, k)$$
$$= 4 e^{p^2 - (p+k)^2 - k^2} p^2 (p+k)_\mu + 4 e^{(p+k)^2 - p^2 - k^2} (p+k)^2 p_\mu$$

- Analytic solution for $p_\mu, k_\mu$

$$\left( F(x) := \frac{e^x - 1}{x} \right)$$

$$V_\mu(p, k) = c_1 p_\mu + c_2 k_\mu$$
$$+ 2F\left(p^2 - (p+k)^2 - k^2\right) p^2 (p+k)_\mu + 2F\left((p+k)^2 - p^2 - k^2\right)(p+k)^2 p_\mu$$

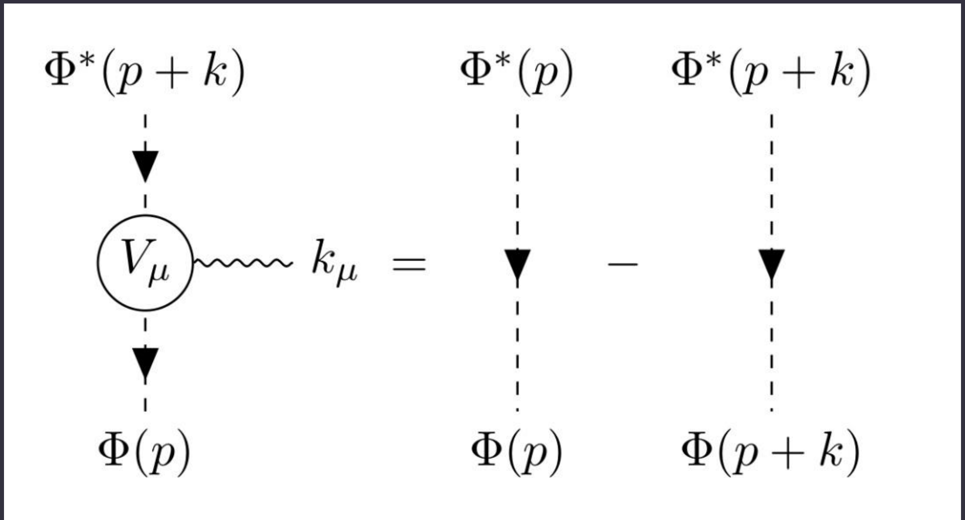
- Integral constants cannot be determined from RG eq.
  ⇒ constrained by Ward-Takahashi identity

# WT id. for 3-pt. function

- Ward-Takahashi identity

$$k_\mu V_\mu(p,k) = e^{(p+k)^2-p^2-k^2} h_S^{-1}(p+k) - e^{p^2-(p+k)^2-k^2} h_S^{-1}(p)$$

$\Rightarrow$ Exact solution $c_1 = 2, c_2 = 1$

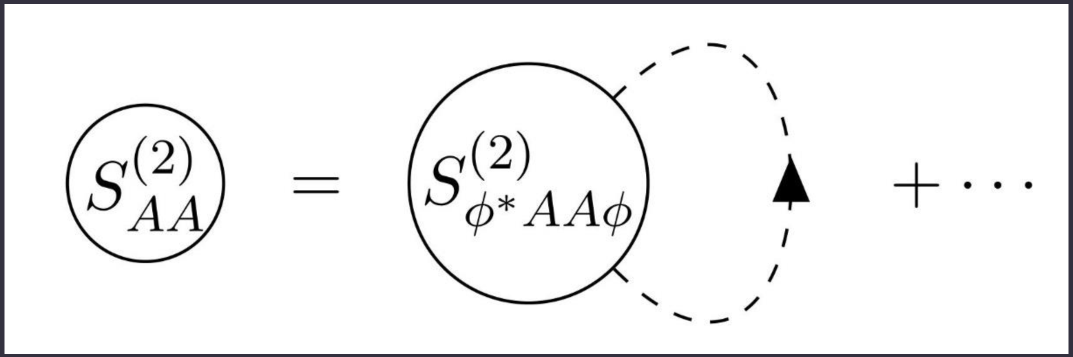


# Second order

- Ansatz

$$S_2 = S_2^{AA} + S_2^{\phi^* AA\phi} + S_2^{\phi^*\phi} + S_2^{\phi^*\phi\phi^*\phi}$$

- We focus on the photon two-point function ($S_2^{AA}$) and the matter-photon four-point function $S^{(2)}_{\phi^* AA\phi}$ ($S_2^{\phi^*\phi}$ and $S_2^{\phi^*\phi\phi^*\phi}$ do not contribute)

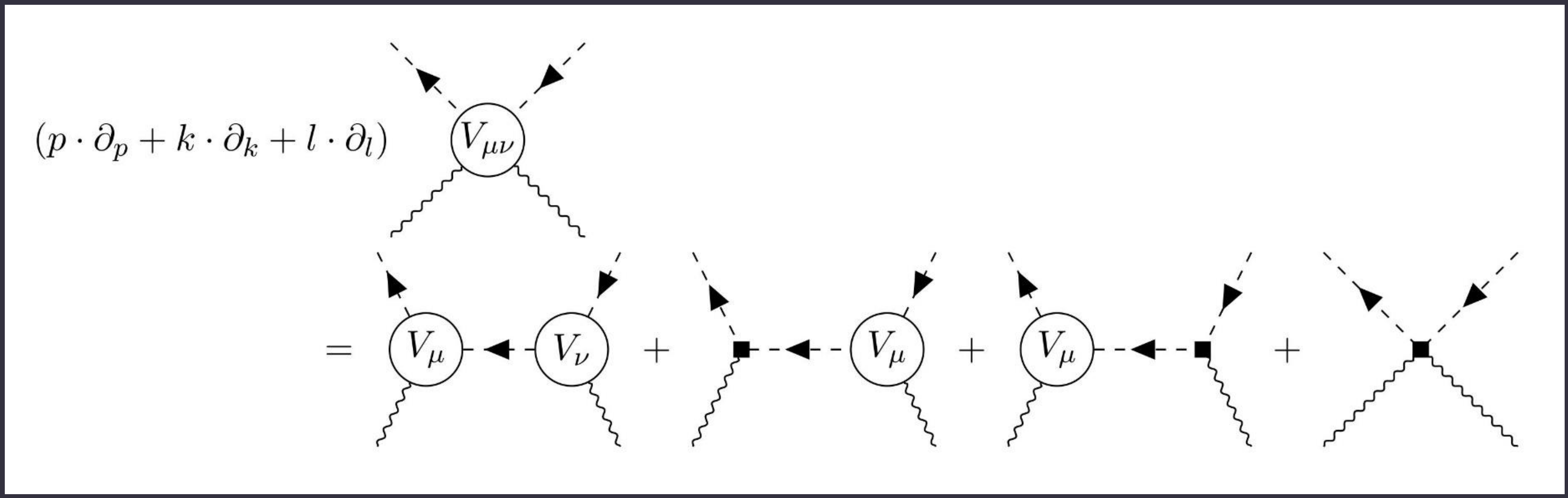


# Four-point function

- Ansatz

$$S^{(2)}_{\phi^* AA\phi} = \int_{p,k,l} V_{\mu\nu}(p,k,l)\Phi^*(p+k+l)\mathcal{A}_\mu(k)\mathcal{A}_\nu(l)\Phi(p)$$

- RG equation



# WT id. for 4-pt. function

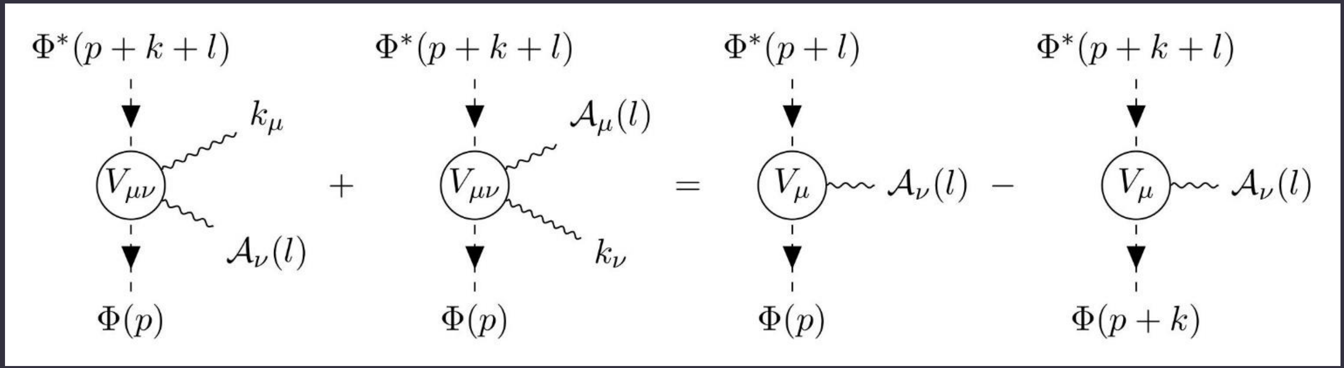
- Analytic solution $(X_{\mu\nu} = O(p^2, k^2, l^2 \dots)$ :solution to some differential eq.)

$$V_{\mu\nu}(p,k,l) = c_3\delta_{\mu\nu} + V_\mu(p,k)h_S(p+k)V_\nu(p+k,l)$$
$$-\delta_{\mu\nu}\left((p+k+l)^2 F\left((p+k+l)^2 - p^2 - k^2 - l^2\right) + p^2 F(p^2 - (p+k+l)^2 - k^2 - l^2)\right)$$
$$-4X_{\mu\nu}(p,k,l)$$

- Ward-Takahashi identity

$$k_\mu\left(V_{\mu\nu}(p,k,l) + V_{\nu\mu}(p,l,k)\right)$$
$$= e^{(p+k+l)^2-(p+l)^2}\frac{h_S(p+l)}{h_S(p+k+l)}V_\nu(p,l) - e^{p^2-(p+k)^2}\frac{h_S(p+k)}{h_S(p)}V_\nu(p+k,l)$$

$\Rightarrow$ Exact solution $c_3 = -1$



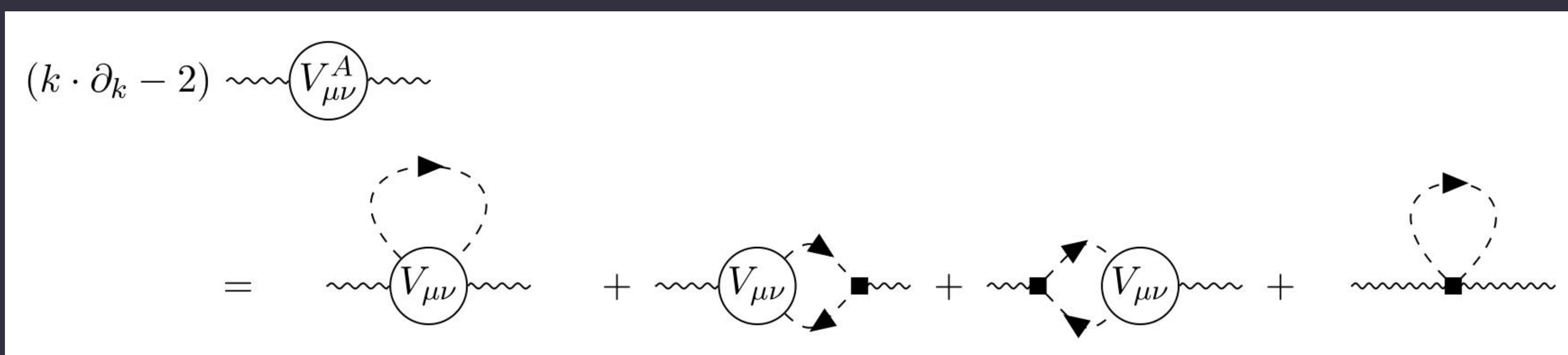
# Photon two-point function

- Ansatz

$$S_{AA}^{(2)} = \frac{1}{2}\int_k V_{\mu\nu}^{A}(k)\mathcal{A}_\mu(k)\mathcal{A}_\nu(-k)$$

- Expand $V_{\mu\nu}(k)$ in terms of $k$
  $\Rightarrow$ zeroth order $\cdots$ mass term
  second order $\cdots$ kinetic term

- RG equation



# Two Point Function at $O(e_\tau^2)$

- Calculated in four dimensions

$$V^A_{\mu\nu}(k) = m_A^2 \delta_{\mu\nu} + g_1\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) + g_2 k_\mu k_\nu + O(k^4)$$

$$m_A^2 = 0, \quad g_1 = 1/48\pi^2, \qquad g_2 = 0$$

$\Rightarrow$ consistent with the perturbation theory

- Result 2:

$m_A^2$ is exactly zero in general dimensions
$\Rightarrow$ GF-ERG gives a gauge-invariant RG flow?

(c.f. $m_A^2 \neq 0$ in the conventional ERG (WP eq.))

# Contents

# Summary

- Applied Gradient Flow Exact Renormalization Group to Scalar Quantum Electrodynamics
- Gave an RG flow based on the diffusion equation consistent with the off-shell BRST transformation
- Showed the modified BRST invariance reduces to the ordinary Ward-Takahashi identity if the ghost sector is free
- Solved the GF-ERG equation perturbatively
  - Consistent results with the perturbation theory
  - Showed the 1-loop contribution to the photon mass vanishes in general dimensions
- Implication: GF-ERG gives a gauge-invariant RG flow

# Discussion

- Consistency with the perturbation theory in the matter sector
  (Inconsistent in QED case, but consistent with those of the gradient-flowed field)

- Application of GF-ERG to quantum gravity
  - Can we define an RG flow in a manifestly diffeomorphism-invariant way?

Thank you!