Gauge Symmetry and Functional RG

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Katsumi Itoh

Ed. Dept. Niigata Univ.

Review, Igarashi, KI and Sonoda, Progress of Theoretical Physics Supplement No. **181** (2009);
Igarashi, KI and Pawlowski (2016); Igarashi, KI and Morris, PTEP (2019); Igarashi and KI, PTEP (2021);
Echigo, Igarashi, KI, Pawlowski and Yu Takahashi, FRG flow in QED (to appear)

Today's talk

In the presence of a momentum cutoff, gauge symmetry is realized in a modified form.

The Batalin-Vilkovisky's antifield formalism, the BV formalism, is appropriate to describe the modifed gauge symmetry.

In the BV formalism, the quantum master equation (QME) guarantee the presence of the modified gauge symmetry.

We consider gauge theory with fRG based on the Batalin-Vilkovisky's antifield formalism.

The two key equations are the flow equation and quantum master equation (QME).

It will be explained:

- Two equations, the flow eq. and QME, are formally compatible;

- Two eqs. can be simultaneously solved to give a perturbative action;

- Based on our perturbative results, we present a “gauge-consistent” fRG flows for QED with chirally invariant four-fermi interactions. Remaining problems are pointed out.

Further points

We will find that:

- It is important to keep QME along the RG flow to realize gauge symmetry — We introduce the wave function renormalzation to keep the canonical structure of the Batalin and Vilkovisky formalism.

- QME is equivalent to BRST symmetry and its nilpotency. — The nilpotency is necessary to realize the structure of the BRST algebra (cf. Kugo-Ojima formalism).

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The anti-field formalism a la Batalin-Vilkovisky

For a classical gauge fixed action $S_0[\phi]$ for a generic gauge theory, define an extended action as

$$S_{\rm cl}[\phi,\phi^*] \equiv S_0[\phi] + \phi^*_A \delta\phi^A$$

- Here antifields ϕ^*_A are introduced as sources for the BRST transformations $\delta\phi^A$.

the canonical structure via the antibracket for any field variables X and Y, we define

$$(X,Y) \equiv \frac{\partial^r X}{\partial\phi^A}\frac{\partial^l Y}{\partial\phi^*_A} - \frac{\partial^r X}{\partial\phi^*_A}\frac{\partial^l Y}{\partial\phi^A}$$

$$(S_{\rm cl}, S_{\rm cl}) = 2(\delta S_0 + \phi^*_A\delta^2\phi^A)$$

Classical master equation (CME): $(S_{\rm cl}, S_{\rm cl}) = 0 \Leftrightarrow$ action invariance *and* the nilpotency.

Generalize the consideration for $S[\phi, \phi^*]$ that defines a quantum system via the functional integration over ϕ.

$$\int D\phi \; e^{-S[\phi,\phi^*]}$$

Under the BRST transformation of fields $\delta\phi^A \equiv (\phi^A, S) = \frac{\partial^l S}{\partial\phi^*_A}$, the changes of the action and the functional measure are summed up to **the quantum master operator**:

$$\Sigma[\phi, \phi^*] \equiv \frac{\partial^r S}{\partial\phi^A}\frac{\partial^l S}{\partial\phi^*_A} - \frac{\partial^r}{\partial\phi^A}\delta\phi^A = \frac{1}{2}(S,\ S) - \Delta S\,, \qquad \Delta \equiv (-)^{\epsilon_A+1}\frac{\partial^r}{\partial\phi^A}\frac{\partial^r}{\partial\phi^*_A}$$

where $\epsilon_A \equiv \epsilon(\phi^A)$ is the Grassmann parity.

The system is BRST invariant quantum mechanically if the two contributions cancel:

$$\Sigma[\phi, \phi^*] = 0\ . \quad \text{(QME)}$$

The quantum BRST transformation as

$$\delta_Q X \equiv (X, S) - \Delta X$$

We have two important algebraic identities without assuming QME:

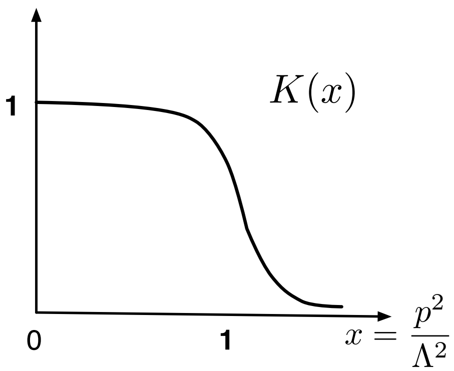
$$\begin{aligned}\delta_Q \Sigma[\phi, \phi^*] &= 0\,,\\ \delta_Q^2 X &= (X, \Sigma[\phi, \phi^*])\,.\end{aligned}$$

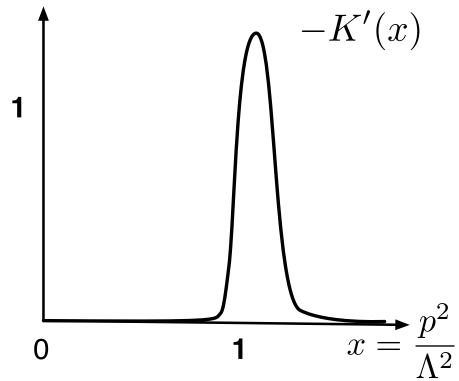
The quantum BRST transformation is nilpotent if and only if QME holds.

$\Sigma[\phi, \phi^* = 0] = 0$ is the WT identity.

Wilson action via path integral with a momentum cutoff Λ

$$Z_\Lambda = \int [d\phi]\, e^{-S_\Lambda[\phi]}, \qquad S_\Lambda[\phi] = \frac{1}{2}\int_p \frac{\Delta^{-1}_{AB}(p)}{K(p/\Lambda)} \phi^A(p)\phi^B(-p) + S_{I,\Lambda}[\phi]\,.$$





Polchinski eq., the flow equation $\quad t = \ln \Lambda/\mu$

$$\dot{S}_{I,\Lambda}[\phi] = \partial_t S_{I,\Lambda}[\phi] = \frac{1}{2}\int_p \dot{K}(p/\Lambda)\cdot \Delta^{AB}(p)\Big(\frac{\partial S_{I,\Lambda}}{\partial\phi^A(p)}\frac{\partial S_{I,\Lambda}}{\partial\phi^B(-p)} - \frac{\partial^2 S_{I,\Lambda}}{\partial\phi^A(p)\partial\phi^B(-p)}\Big)$$

From the shape of $\dot{K}(p/\Lambda)$, it is easy to understand that the momentum integration on the r.h.s. is taken only around $p^2 \sim \Lambda^2$ and it is finite. There is no infinity one might expect in a FT calculation.

QME for Wilson action and its flow

Appropriate to consider gauge symmetry in the functional renormalization group with BV formalism.

$$\Sigma_\Lambda[\phi,\phi^*] \equiv \frac{\partial^r S_\Lambda}{\partial\phi^A} K \frac{\partial^l S_\Lambda}{\partial\phi^*_A} - \frac{\partial^r}{\partial\phi^A}\delta\phi^A = \frac{1}{2}(S_\Lambda, S_\Lambda)_K - \Delta_K S_\Lambda\,,$$

$$\Delta_K \equiv (-)^{\epsilon_A+1} K \frac{\partial^r}{\partial\phi^A}\frac{\partial^r}{\partial\phi^*_A}\,.$$

$\Sigma_\Lambda = 0$ implies the presence of gauge symmetry.

Under the scale change, Σ_Λ behaves as

$$\dot\Sigma_\Lambda = \left[\int_p \dot K(p/\Lambda)\Delta^{AB}(p)\left(\frac{\partial S_{I,\Lambda}}{\partial\phi^A}\frac{\partial}{\partial\phi^B} - \frac{1}{2}\frac{\partial^2}{\partial\phi^A\partial\phi^B}\right)\right]\Sigma_\Lambda$$

Once $\Sigma_\Lambda = 0$ at some scale, Σ_Λ vanishes along the RG flow.

Properties of Wilsonian and 1PI actions

	Wilsonian action	1PI action
	$S_\Lambda[\phi, \phi^*]$	$\Gamma_\Lambda[\Phi, \Phi^*]$
Diagrams	Connected	1PI
Gauge symmetry	mod. WT id.	mod. WT id.
RG flow eq.	Polchinski eq.	1PI flow eq.

Two actions are related via a Legendre transformation.

$$\Gamma_{I,\Lambda}[\Phi, \Phi^*] = S_{I,\Lambda}[\phi, \phi^*] + \frac{1}{2}(\Phi - \phi)^A \bar{\Delta}^{-1}_{AB}(\Phi - \phi)^B$$

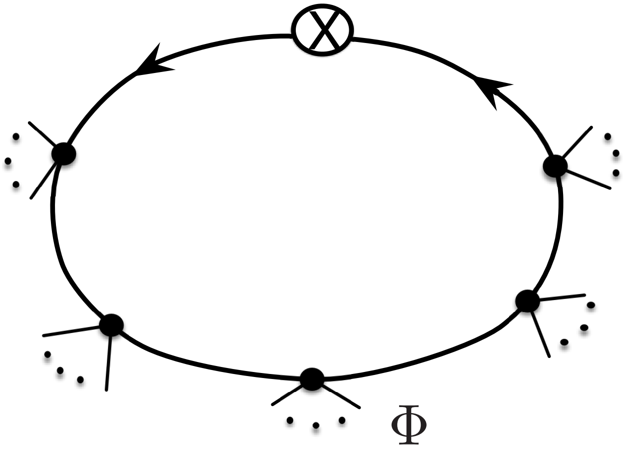
$$(\Phi - \phi)^A = \bar{\Delta}^{AB}\frac{\partial^l S_{I,\Lambda}}{\partial \phi^B} = \bar{\Delta}^{AB}\frac{\partial^l \Gamma_{I,\Lambda}}{\partial \Phi^B}$$

where $\bar{\Delta} \equiv (1-K)\Delta = \bar{K}\Delta$ is the high momentum propagator that allows momentum modes above the cutoff Λ to propagate.

Legendre tr. of flow eq. for $S_I \Rightarrow$ 1PI flow eq. for Γ_I $\qquad t = \ln \Lambda/\mu$

From now on, we drop the subscript Λ.

$$\dot{\Gamma}_I = -\frac{1}{2}\mathrm{Str}\left(\dot{\bar{\Delta}}\bar{\Delta}^{-1}\left[1 + \bar{\Delta}\Gamma_I^{(2)}\right]^{-1}\right) = -\frac{1}{2}\mathrm{Str}\left(\bar{\Delta}^{-1}\dot{\bar{\Delta}}\bar{\Delta}^{-1}\left[\bar{\Delta}^{-1} + \Gamma_I^{(2)}\right]^{-1}\right)$$

$$\dot{\Gamma}_I = -\frac{1}{2}$$ 

- $\otimes$ represents $\bar{\Delta}^{-1}\dot{\bar{\Delta}}\bar{\Delta}^{-1} = -\frac{d}{dt}(\bar{K}(p/\Lambda))^{-1} \cdot \Delta^{-1} = -\dot{K}\bar{K}^{-2}\Delta^{-1}$.
- The full propagator $[\bar{\Delta}^{-1} + \Gamma_I^{(2)}]^{-1}$ is shown as the arrowed line:
  - Blobs on the line are vertices; small dots are for fields Φ and Φ^*.

QME in terms of 1PI action

Rewrite the QME, $\Sigma = (S, S) - \Delta S/2 = 0$, in terms of 1PI action.
The Legendre tr. of QME gives the modified Ward-Takahashi id. [1](Ellwanger):

$$\Sigma = \frac{1}{2}(\Gamma, \Gamma) - \mathrm{Str}\left(K\Gamma_{I*}^{(2)} \left[1 + \bar{\Delta}\Gamma_I^{(2)} \right]^{-1} \right) = 0\,,$$

where

$$\left(\Gamma_{I*}^{(2)}\right)^A{}_B = \frac{\partial^l \partial^r}{\partial \Phi_A^* \partial \Phi^B} \Gamma_I\,, \qquad \left(\Gamma_I^{(2)}\right)_{AB} = \frac{\partial^l \partial^r}{\partial \Phi^A \partial \Phi^B} \Gamma_I\,.$$

The measure term is a one-loop structure with the same full propagator $\bar{\Delta}^{-1} + \Gamma_I^{(2)}$ as the flow eq. but with the different vertex $K\Gamma_I^{(2)}\bar{\Delta}^{-1}$.

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[1]We can show

$$\because \quad (S, S)_K = (\Gamma, \Gamma), \quad \Delta S_I = \mathrm{Tr}\left(K\Gamma_{I*}^{(2)} \left[1 + \bar{\Delta}\Gamma_I^{(2)} \right]^{-1} \right)\,.$$

Simultaneous solutions to the mod. WT id. and the flow equation

$$\Sigma = \frac{1}{2}(\Gamma, \Gamma) - \mathrm{Str}\left(K\Gamma_{I*}^{(2)} \left[1 + \bar{\Delta}\Gamma_I^{(2)} \right]^{-1} \right) = 0$$

$$\dot{\Gamma}_I = -\frac{1}{2}\mathrm{Str}\left(\dot{\bar{\Delta}}\bar{\Delta}^{-1} \left[1 + \bar{\Delta}\Gamma_I^{(2)} \right]^{-1} \right)$$

Construction of perturbative solutions for Yang-Mills (Igarashi, Itoh, Morris) and QED (Igarashi, Itoh).

An important lesson: introduce the Z factors so that BV structure is intact.

Perturbative compatibility of Flow eq. and QME

(Igarashi, KI and Morris (2019) for YM, Igarashi and KI (2020) for QED)

- Flow eq. and QME may be solved perturbatively.
  — Of course, this is related to the known results on the perturbative renormalizability. (cf. The PTP suppl. review)

- In the methods in the above papers, the Wilsonian as well as 1PI actions may be constructed explicitly even in the presence of a finite cutoff.

- Though our result applies to YM as well, today we consider QED with the chiral invariant 4-fermi interactions.

Construct an action based on the BRST algebra

The quantum BRST transformation

$$\delta_Q X = (\mathcal{Q} + \mathcal{Q}^- - \Delta) X$$

$$\mathcal{Q}\Phi^A = (\Phi^A, \Gamma) = \frac{\partial^l \Gamma}{\partial \Phi^*_A}, \qquad \mathcal{Q}^- \Phi^*_A = (\Phi^*_A, S) = -\frac{\partial^l \Gamma}{\partial \Phi_A}$$

$\mathcal{Q}^-$ is called **the Kozsul-Tate operator** that changes the anti-ghost number.

(Formalism developed by Fisch and Henneaux (1990) and others.)

	G-parity	gh #	anti-gh #	dimension
A_μ	0	0	0	1
C	1	1	0	1
$\Psi,\ \bar{\Psi}$	1	0	0	$3/2$
$\Psi^*,\ \bar{\Psi}^*$	0	-1	1	$3/2$
$\bar{C}$	1	-1	1	1
B	0	0	1	2
A^*_μ	1	-1	1	2
$\bar{C}^*$	0	0	0	2

Expansion in terms of e : $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \cdots$

The classical action $\Gamma_{\rm cl} = \Gamma_0 + \Gamma_1 + \Gamma_{2,\rm cl}$,

$$
\begin{aligned}
\Gamma_0 &= \int_x \left[\frac{1}{2}\left\{ (\partial_\mu A_\nu)^2 - (\partial \cdot A)^2 \right\} + \bar{\Psi} i \partial\!\!\!/ \Psi + \left(A_\mu^* - i\partial_\mu \bar{C} \right) \partial_\mu C + \frac{1}{2}\xi B^2 + \left(\bar{C}^* - i\partial \cdot A \right) B \right], \\
\Gamma_1 &= \int_x \left[-e\bar{\Psi} A\!\!\!/ \Psi - ie\Psi^* \Psi C + ie\bar{\Psi}\bar{\Psi}^* C \right], \\
\Gamma_{2,\rm cl} &= \int_x \left[\frac{G_S}{2\Lambda^2} \left\{ \left(\bar{\Psi}\Psi\right)\left(\bar{\Psi}\Psi\right) - \left(\bar{\Psi}\gamma_5\Psi\right)\left(\bar{\Psi}\gamma_5\Psi\right) \right\} \right. \\
&\qquad \left. + \frac{G_V}{2\Lambda^2} \left\{ \left(\bar{\Psi}\gamma_\mu\Psi\right)\left(\bar{\Psi}\gamma_\mu\Psi\right) + \left(\bar{\Psi}\gamma_5\gamma_\mu\Psi\right)\left(\bar{\Psi}\gamma_5\gamma_\mu\Psi\right) \right\} \right] ,
\end{aligned}
$$

satisfies the Classical Master Equation (CME) up to $\mathcal{O}(e^2)$,

$$
(\Gamma_{\rm cl}, \Gamma_{\rm cl}) = 0 \,.
$$

- "$\mathrm{Str}\log$-formula" for one-loop effective action

$$\Gamma_q = \frac{1}{2}\mathrm{Str}\log\left(\bar{\Delta}^{-1} + \Gamma^{(2)}_{I,\mathrm{cl}}\right) ,$$

which satisfies the flow eq. at lowest order in coupling !
$\Gamma^{(2)}_{I,\mathrm{cl}}$ are field dependent and give rise the diagrams given below.

Contributions to Γ_q:

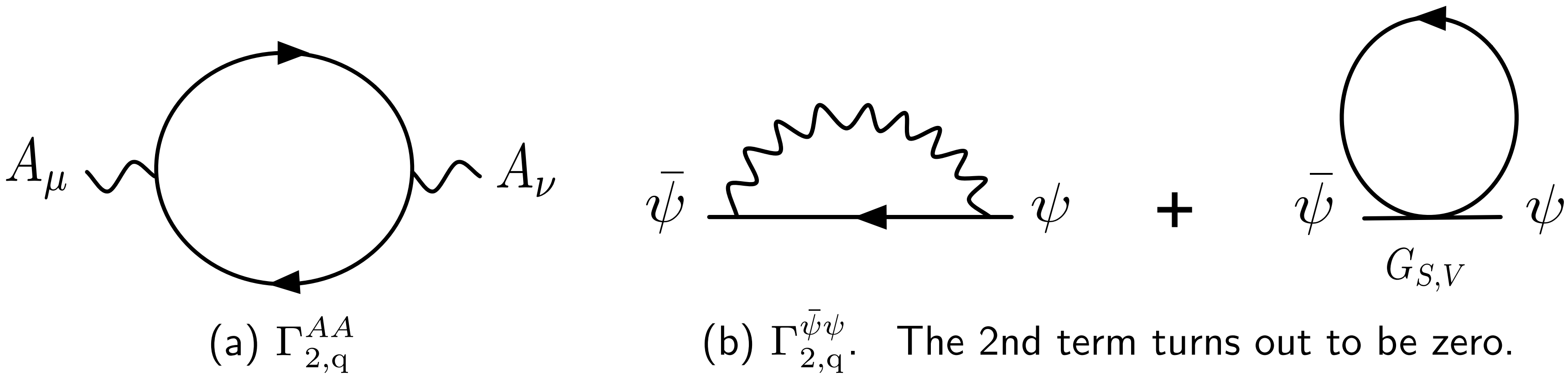


Figure 1: Contributions to $\Gamma_{2,\mathrm{q}}$. Interaction vertices consist of vertices and fields with no external lines. **Internal lines are IR-regularized propagators $\bar{\Delta}$.**

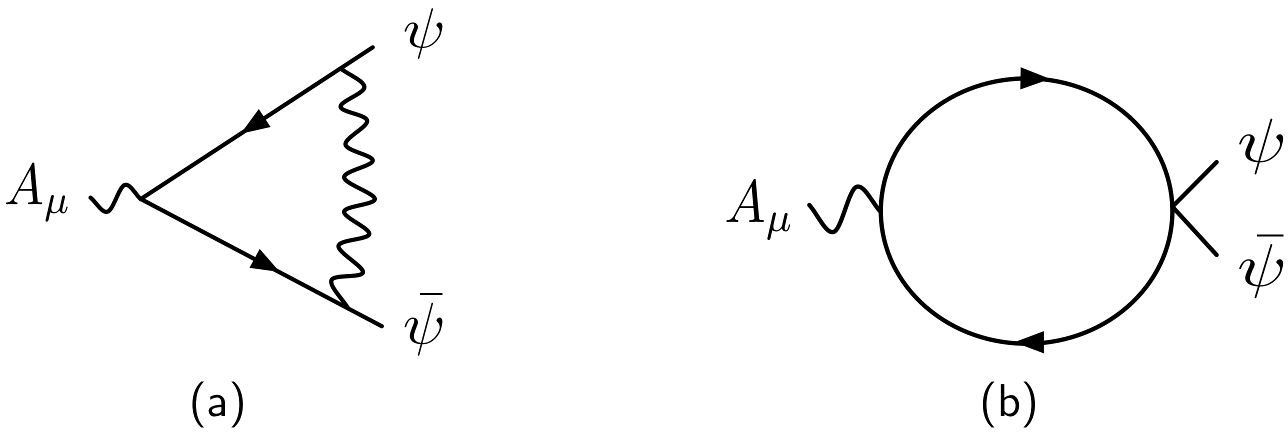


Figure 2: Contributions to $\Gamma_{3,\mathrm{q}}$. Interaction vertices consist of vertices and fields with no external lines. Internal lines are IR-regularized propagators $\bar{\Delta}$.

One-loop correction to gauge two-point function Γ_q^{AA} and the anomalous dimention η_A

For the photon two-point functions,

$$\dot{\Gamma}_{\mathrm{q}}^{AA} = e^2 \int_{p,q} \mathrm{Tr}\left[\frac{\dot{\bar{K}}(q)}{\not{q}} \not{A}(p) \frac{\bar{K}(p+q)}{(\not{p}+\not{q})} \not{A}(p)\right]$$
$$= \frac{e^2}{2} \int_p A_\mu(-p) \dot{\mathcal{A}}_{\mu\nu}(p) A_\nu(p)\,.$$

By expanding $\dot{\mathcal{A}}_{\mu\nu}(p)$ in the external momentum up to $\mathcal{O}(p^2)$,
$\dot{\mathcal{A}}_{\mu\nu}(p) = 2M_A^2 \delta_{\mu\nu} - \eta_A (p^2 \delta_{\mu\nu} - p_\mu p_\nu) + \cdots,$
we find the photon mass term and the anomalous dimension as

$$M_A^2 = \Lambda^2 \cdot \frac{e^2}{4\pi^2} \int_0^\infty du\; u\bar{K}'(u)\bar{K}(u)\,, \qquad \eta_A = \frac{e^2}{6\pi^2}\,.$$

The Ward identity, $Z_1 = Z_2$, is modified in the presence of 4-fermi couplings

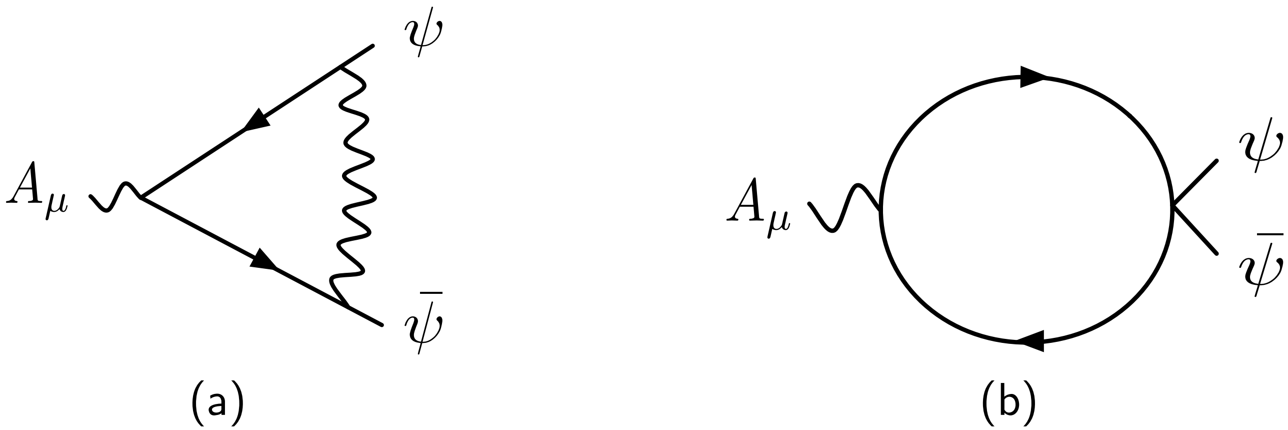


Figure 3: Contributions to $\Gamma_{3,\mathrm{q}}$. Interaction vertices consist of vertices and fields with no external lines. Internal lines are IR-regularized propagators $\bar{\Delta}$.

For the running of e, the scale derivatives of the above one-loop diagarams to give

$$\eta_e = \frac{1}{2}\eta_A - \frac{1}{4\pi^2}(G_S - 4G_V)\int_0^\infty du\ u\bar{K}(u)'\bar{K}(u)\,.$$

where the anomalous dimensions η's defined as

$$Z_{3,2} = 1 - \eta_{A,\Psi} \log(\Lambda/\mu)\,,$$
$$Z_e = 1 + \eta_e \log(\Lambda/\mu)\,.$$

This implies that

$$Z_e Z_3^{1/2} \neq 1\,, \quad \text{or} \quad Z_1 \neq Z_2\,.$$

QME

- The above obtained action $\Gamma \equiv \Gamma_{\rm cl} + \Gamma_{\rm q}$ satisfies the QME

$$\Sigma = \frac{1}{2}(\Gamma, \Gamma) - \mathrm{Str}\left(K\Gamma^{(2)}_{I*} \left[1 + \bar{\Delta}\Gamma^{(2)}_{I} \right]^{-1} \right) = 0$$

up to the order $\mathcal{O}(e^3)$.

- Expansion of QME $\Sigma = 0$

We have the phton A_μ, a fermion $\bar{\psi}, \psi$ and ghosts and their antifields. We may expand Σ in terms of fields and antifields. We consider the first two simple combinations of fields,

$$0 = \Sigma = \Sigma^{AC} + \Sigma^{\bar{\psi}\psi C} + \cdots$$

• The condition $\Sigma^{AC}=0$ gives the longitudinal part of quantum correction to the photon 2pt function.

$$\Sigma^{AC}=(A^*_\mu\partial_\mu C,\ \ \Gamma_{\rm q}^{AA})-{\rm Str}\left(K\Gamma^{(2)}_{I*}\left[1+\bar\Delta\Gamma^{(2)}_I\right]^{-1}\right)^{AC}=0$$

In the first term, we find the longitudinal part that balance with the measure contribution. This way we obtain $\mathcal{L}$.

• Schematically, $\Sigma^{\bar\psi\psi C}=0$ may be expressed in a similar manner as

$$\begin{aligned}\Sigma^{\bar\psi\psi C} &= (A^*_\mu\partial_\mu C,\ \ \Gamma_{\rm q}^{\bar\psi A\psi})+(e\psi^*\psi C+e\bar\psi^*\bar\psi C,\ \Gamma_{\rm q}^{\bar\psi\psi}) \\ &\quad -{\rm Str}\left(K\Gamma^{(2)}_{I*}\left[1+\bar\Delta\Gamma^{(2)}_I\right]^{-1}\right)^{\bar\psi\psi C}=0\end{aligned}$$

We are familiar with the first two terms. Here we have the extra measure term.

Dimensionless formulation

- Easy to count mass dimensions for the quantities in the flow equation.

- We know that there could appear logarithmic terms like $\ln(\Lambda/\mu)$ as we have seen in a perturbative calculation. The wave function renormalization is of this type.

- $\bar{x}_\mu = \Lambda x_\mu$ and $\bar{p}_\mu = p_\mu/\Lambda$, $\delta^d(\bar{p}) = \Lambda^d \delta^d(p)$.

- The dimensionless fields

$$
\begin{aligned}
\bar{\Phi}^A(\bar{x}) &= \sqrt{Z_A}\ \Lambda^{-d_A}\ \Phi(x)\,, \\
\bar{\Phi}^A(\bar{p}) &= \sqrt{Z_A}\ \Lambda^{d-d_A}\ \Phi(p)\,.
\end{aligned}
$$

1. Define coefficients by expanding Γ_Λ in terms of fields

2. Replace all the quantities by their dimensionless forms

3. We may define the dimensionless coefficients

$$\Gamma_\Lambda = \sum_{n=2}^{\infty} \int \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_n}{(2\pi)^d} (2\pi)^d \delta^d(p_1 + \cdots p_n) \Gamma_{A_1,\cdots,A_n}(\Lambda;\ p_1,\cdots,p_n) \Pi_{i=1}^n \Phi^{A_i}(p_i)$$

$$= \sum_{n=2}^{\infty} \int_{\bar{p}_i} \Lambda^{nd} \Lambda^{-d} (2\pi)^d \delta^d(\bar{p}_1 + \cdots \bar{p}_n) \Gamma_{A_1,\cdots,A_n}(\Lambda;\ p_1,\cdots,p_n) \Pi_{i=1}^n \Lambda^{-d+d_{A_i}} \frac{\bar{\Phi}^{A_i}(\bar{p}_i)}{\sqrt{Z_{A_i}}}\,.$$

Define the dimensionless coefficients $\bar{\Gamma}_{A_1,\cdots,A_n}$

$$\bar{\Gamma}_{A_1,\cdots,A_n}(t; \bar{p}_1,\cdots,\bar{p}_n) \equiv \frac{\Lambda^{\sum_i d_{A_i} - d}}{\sqrt{Z_{A_1}\cdots Z_{A_n}}}\, \Gamma_{A_1,\cdots,A_n}(\Lambda;\ p_1,\cdots,p_n)\,.$$

Dimensionless flow equation[2]

Finally, we reach the dimensionless flow equation

$$\dot{\Gamma}_I[\Phi] = -(-)^{\epsilon_A}\frac{1}{2}\int_p r_A(p)\Delta^{-1}_{AB}(p)\left[\left(\bar{\Delta}^{-1} + \Gamma_I^{(2)}\right)^{-1}\right]^{AB}(p,-p)$$
$$-d\;\Gamma_\Lambda[\Phi] + (d_A + \eta_A/2)\int_p \Phi^A(p)\frac{\partial^l \Gamma_\Lambda[\Phi]}{\partial \Phi^A(p)} + \int_p \Phi^A(p)\;p\cdot\frac{\partial}{\partial p}\left(\frac{\partial^l}{\partial\Phi^A(p)}\right)'\Gamma_\Lambda[\Phi]\;,$$

where $-r_A\bar{\Delta}^{-1}$ corresponds to $\otimes$ with

$$r_A(p) = -\partial_t\left(\frac{K}{1-K}\right) + \eta_A\frac{K}{1-K} = \frac{2x\;K'(x)}{\left(1-K(x)\right)^2} + \eta_A\frac{K(x)}{1-K(x)}\;.$$

- Expanding the dimensionless flow equation in terms of fields, we find a set of differential equations for the dimensionless couplings.

[2] The bar for dimensionless fields are omitted.

Ansatz to 1PI action for dimensionless flow eq.

(Here we have removed the antif-field dependence.)

$$
\begin{aligned}
\Gamma[\Phi] &= \frac{1}{2}\Phi^A {\Delta^{-1}}_{AB}\Phi^B + \Gamma_I[\Phi] \\
&= \frac{1}{2}\int_p A_\mu(-p)\Big[P^T_{\mu\nu}\Big\{p^2 + \mathcal{T}(t,p^2)\Big\} + P^L_{\mu\nu}\Big\{\xi^{-1}p^2 + \mathcal{L}(t,p^2)\Big\}\Big]A_\nu(p) \\
&+ \int_p \left[\bar{C}(-p)ip^2C(p) + \bar{\Psi}(-p)\not{p}\Psi(p)\right] \\
&- e(t)\int_{p,q}\bar{\Psi}(-p)\not{A}_\mu(p-q)\Psi(q) + \frac{1}{2}\int_{p_1,\cdots,p_4}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\
&\times\Big[G_S(t)\Big\{\left(\bar{\Psi}(p_1)\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\Psi(p_4)\right) - \left(\bar{\Psi}(p_1)\gamma_5\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\gamma_5\Psi(p_4)\right)\Big\} \\
&+ G_V(t)\Big\{\left(\bar{\Psi}(p_1)\gamma_\mu\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\gamma_\mu\Psi(p_4)\right) + \left(\bar{\Psi}(p_1)\gamma_5\gamma_\mu\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\gamma_5\gamma_\mu\Psi(p_4)\right)\Big\}\Big]
\end{aligned}
$$

- $\mathcal{T}$ and $\mathcal{L}$ are quantum corrections to the gauge two point functions and contribute to $\Gamma^{(2)}$ and they appear in the nominator of the photon propagator.

Flow equation for photon two-point part of 1PI action, $\mathcal{T}$ and $\mathcal{L}$

$\mathcal{T}$ and $\mathcal{L}$ are functions of $x = p^2$ and $t = \ln \Lambda/\mu$.

By using the notation with $\alpha(t) = e^2(t)$

$$D_x \equiv x\frac{\partial}{\partial x} - 1, \quad D_t \equiv \frac{\partial}{\partial t} - \eta_A(\alpha(t), \xi(t)) \,,$$

the flow equations are

$$(D_x - \tfrac{1}{2}D_t)\mathcal{T}(t,x) + \tfrac{\eta_A}{2}x = \alpha\ C_T^{(0)}(x) + \alpha\eta_\psi\ C_T^{(1)}(x) \,,$$
$$(D_x - \tfrac{1}{2}D_t)\mathcal{L}(t,x) - \tfrac{x}{2}D_t\xi(t)^{-1} = \alpha\ C_L^{(0)}(x) + \alpha\eta_\psi\ C_L^{(1)}(x) \,,$$

- $\alpha(t)$ and $\eta_{A,\psi}(\alpha(t), \xi(t))$ are t-dependent coefficients.
- $C_{T,L}^{(i)}$ are coefficients functions of $x = p^2$.
- Extracting x-linear terms from flow eqs., we find:

- Anomalous dimentions η_A are determined algebraically.
- Flow eq. of ξ, the gauge parameter, $D_t \xi(t)^{-1} = 0$, or

$$\partial_t \xi = -\eta_A \xi$$

The Landau gauge $\xi = 0$ is consistent with the flow of ξ.

- The rest are differential equations for $\mathcal{T}(t, x = p^2)$ and $\mathcal{L}(t, x)$.

Choosing the regulator function as $K(x) = \exp(-x)$,
we find differential equations for $\mathcal{T}$, $\mathcal{L}$ in the lowest order in α,

$$(x\partial_x - 1)\mathcal{T}(t,x) = -\frac{\alpha(t)}{8\pi^2 x^2}\left\{4 + \frac{2x^3}{3} - \left(4 + 2x - x^2\right)\exp(-x/2)\right\},$$

$$(x\partial_x - 1)\mathcal{L}(t,x) = \frac{\alpha(t)}{8\pi^2 x^2}\left\{12 - 8x - \left(12 - 2x - x^2\right)\exp(-x/2)\right\},$$

where $\alpha \equiv e^2$. **The x-linear terms in $\mathcal{T}$ and $\mathcal{L}$ are integration constants.**

These differential equations can be solved analytically to give

$$\mathcal{T}(t,x) = \frac{\alpha(t)}{6\pi^2 x^2}\left[1 - \left(1 + \frac{x}{2} - x^2\right)\exp(-x/2) + \frac{x^3}{2}\int_0^x \frac{e^{-u/2}-1}{u}du\right],$$

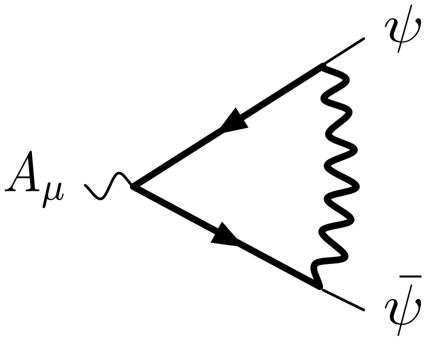
$$\mathcal{L}(t,x) = -\frac{\alpha(t)}{2\pi^2 x^2}\left[1 - x - \left(1 - \frac{x}{2}\right)\exp(-x/2)\right].$$

Two constants of integrations: one is related to the finite amount of wave function renormalisation for the photon field and the other is fixed by comparing the functional forms of $\mathcal{L}(t,x)$ here and that obtained from the WT identity: $\Sigma^{AC} = 0$ gives the same function for $\mathcal{L}$ except a constant.

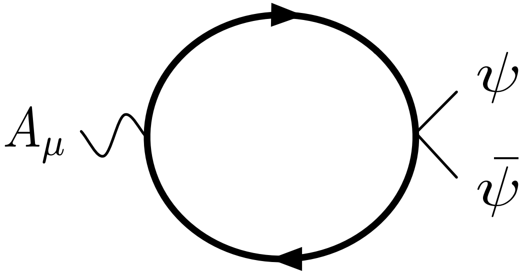
The gauge mass term

Both $\mathcal{T}$ and $\mathcal{L}$ produce a constant term for small $x = p^2$, $3\alpha/16\pi^2$,
that will be a gauge mass $(3\alpha/16\pi^2)\Lambda^2$ once the dimensionality is recovered.
The result is consistend with earlier expression in terms of K.

Flow equations for $\alpha = e^2,\ G_S,\ G_V$: $t \equiv \ln\Lambda/\mu$



(a) e^3 contributions



(b) $eG_{S,V}$ contributions

Figure 4: The regulator term should be inserted to an internal line to produce the contributions to $\partial_t\alpha$.

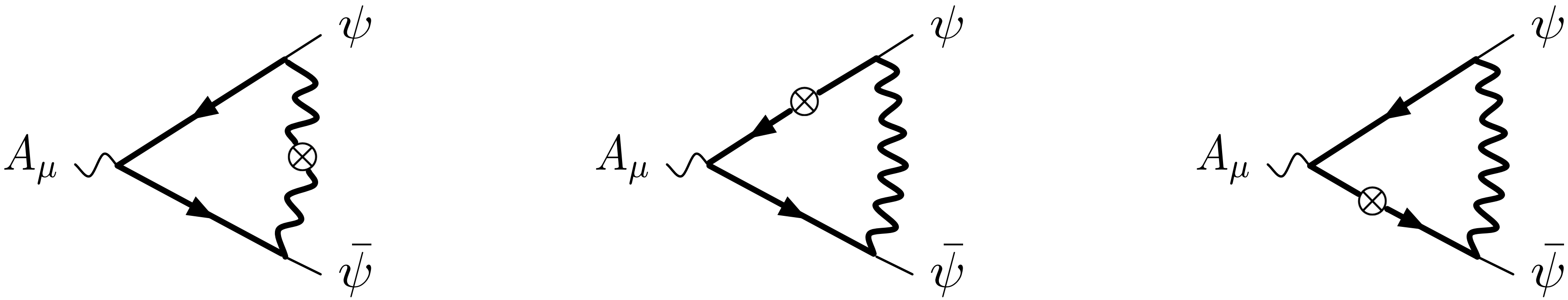
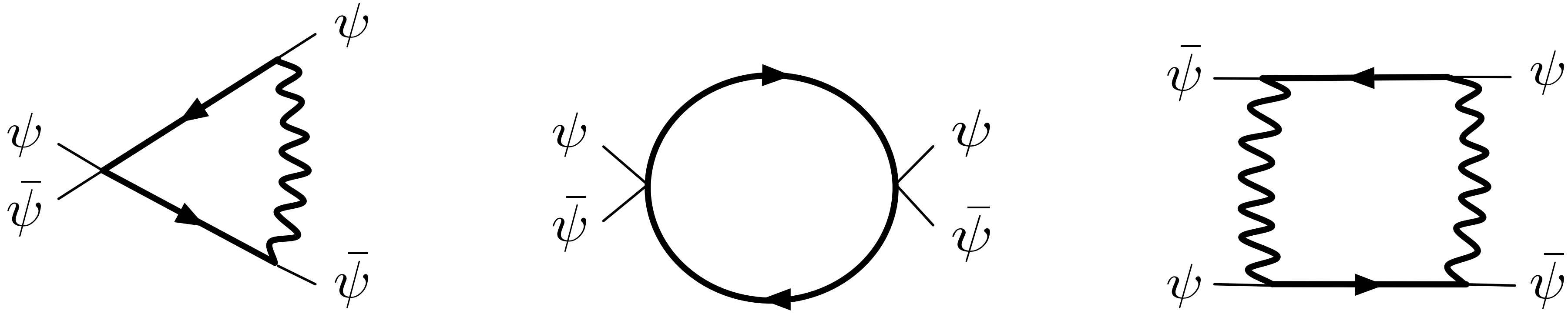


Figure 5: The regulator denoted by $\otimes$ is inserted in each internal line. All three diagrams contribute to the flow equation for α.



(a) $e^2 G_{S,V}$ contributions (b) The 2nd order in $G_{S,V}$ (c) e^4 contributions

Figure 6: The regulator term should be inserted to an internal line to produce the contributions to $\partial_t G_{S,V}$

Flow equations for $\alpha = e^2,\ G_S,\ G_V$: $t \equiv \ln\Lambda/\mu$

$$\frac{\partial \alpha}{\partial t} = (\eta_A + 2\eta_\psi)\alpha - \frac{6\alpha}{(4\pi)^2}(1 - \frac{2}{9}\eta_\psi)(G_S - 4G_V) + 2\alpha^2 \xi I^{(4)}$$

$$\frac{\partial G_S}{\partial t} = 2(1+\eta_\psi)G_S - \frac{3}{(4\pi)^2}\left(1 - \frac{2}{9}\eta_\psi\right)(3G_S - 8G_V)G_S$$
$$+\alpha G_S\left(\eta_A s^{(1)} + \eta_\psi s^{(2)} + s^{(3)}\right) + \alpha^2\left(\eta_A s^{(4)} + \eta_\psi s^{(5)} + s^{(6)}\right)$$

$$\frac{\partial G_V}{\partial t} = 2(1+\eta_\psi)G_V + \frac{3}{2(4\pi)^2}\left(1 - \frac{2}{9}\eta_\psi\right)G_S^2$$
$$+\alpha G_V\left(\eta_A v^{(1)} + \eta_\psi v^{(2)} + v^{(3)}\right) + \alpha^2\left(\eta_A v^{(4)} + \eta_\psi v^{(5)} + v^{(6)}\right)$$

- $I^{(4)},\ s^{(i)},\ v^{(i)}$ are coefficients functions of α and ξ that depend on the regulator function K.

- The anomalous dimensions, $\eta_i \equiv -\partial_t \ln Z_i$ are also functions of α and ξ.

Explicitly, the following are flow equations for the couplings: $r_{A,\psi}$ correspond to $\otimes$.

$$
\begin{aligned}
\partial_t\alpha &= \big(\eta_A + 2\eta_\psi\big)\alpha + \frac{\alpha\big(G_S - 4G_V\big)}{4\pi^2}\int_0^\infty dx\ K\bar{K}^2 r_\psi \\
&\qquad + \frac{\alpha^2}{8\pi^2}\xi\int_0^\infty dx\ K\bar{K}^3\big(x r_A L^2 + 2 r_\psi L\big)\,, \\
\partial_t G_S &= 2(1+\eta_\psi)G_S + \frac{\big(3G_S - 8G_V\big)G_S}{8\pi^2}\int_0^\infty dx K\bar{K}^2 r_\psi \\
&\qquad + \frac{\alpha G_S}{8\pi^2}\int_0^\infty dx K\bar{K}^3\Big(x r_A(3T^2 + \xi L^2) + 2 r_\psi(3T + \xi L)\Big) \\
&\qquad + \frac{3\alpha^2}{8\pi^2}\int_0^\infty dx K\bar{K}^4\big(x r_A T^3 + r_\psi T^2\big)\,, \\
\partial_t G_V &= 2(1+\eta_\psi)G_V - \frac{G_S^2}{16\pi^2}\int_0^\infty dx K\bar{K}^2 r_\psi \\
&\qquad - \frac{\alpha G_V}{8\pi^2}\int_0^\infty dx K\bar{K}^3\Big(x r_A(3T^2 - \xi L^2) + 2 r_\psi(3T - \xi L)\Big) \\
&\qquad + \frac{3\alpha^2}{16\pi^2}\int_0^\infty dx K\bar{K}^4\big(x r_A T^3 + r_\psi T^2\big)\,,
\end{aligned}
$$

Hierarchical Phase Structure: a funnel halved by the $\alpha = 0$ plane.

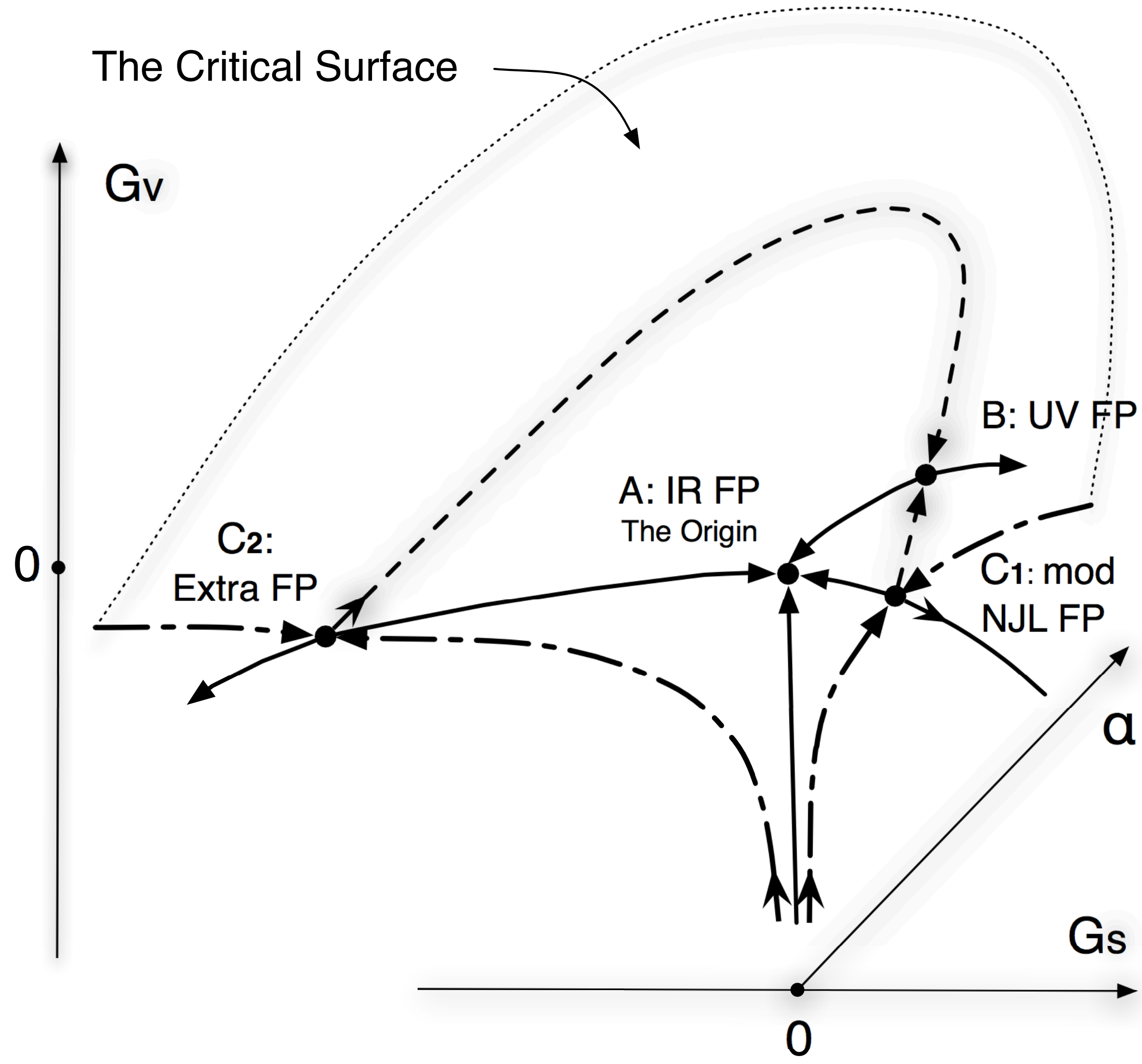


Figure 7: Four fixed points and critical surface and lines

Four fixed points (in Landau gauge)

Fixed Point	# of Rel. ops.	(α, G_S, G_V)
A: I.R.	0	The origin
B: U.V.	1	$(13.5,\ 6.99,\ 0.573)$
C_1: mod. NJL	2	$(0,\ 26.3,\ -3.29)$
C_2: extra	2	$(0, -105,\ -52.6)$

- $(\alpha, G_S, G_V) = (13.8, 8.17, 0.578)$ for the U.V. fixed point in the absence of $\eta_{A,\psi}$, $\mathcal{T}$ and $\mathcal{L}$ in the flow eq.
- Aoki et. al.[3] studied the same system with different regularisation functions and identified the critical coupling without flowing the gauge coupling.
  - Overall phase structure is similar.
  - Some higher order terms are also included here.

[3] PTP **97** (1997) 479. See also Harada et. al. PTP **92** (1994) 1161.

Summary and Discussion: we need to achieve ...

- BV formalism is useful to treat the modified gauge symmetry in ERG.
- We may solve both the flow eq. and QME perturbaively.
- Better understanding is necessary for a non-perturbative study.
- With a numerical analysis, we obtained the phase structure of our model. Further study in needed to support this result.
- The approach based on the homotopy algebra could be helpful (cf. The talk by Matsunaga at Strings and Fields 2022).