A Statistical Mechanics Approach to Holographic RG

Bethe lattice Ising model & p-adic AdS/CFT

Kouichi Okunishi / D02 Group

Tadashi Takayanagi / C01 Group

D02 group

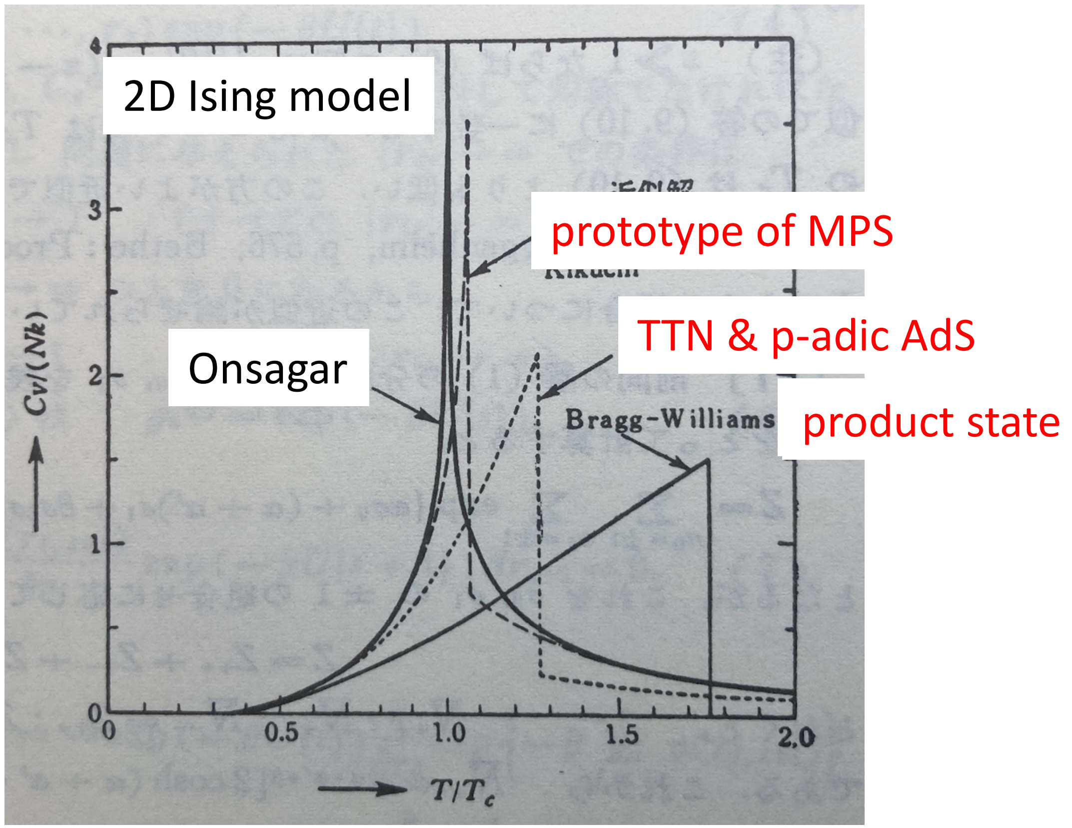
新潟大學
NIIGATA UNIVERSITY

C01 group

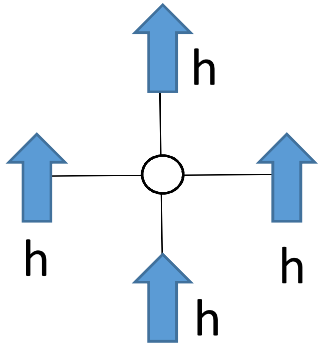
KYOTO UNIVERSITY
KYOTO JAPAN
FOUNDED 1897

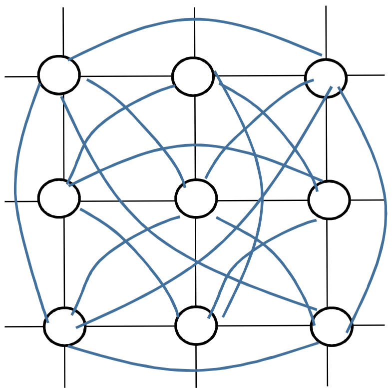
The Bethe lattice Ising model can be served as a prototype toy model for the holographic RG

$\Rightarrow$ p-adic AdS/CFT



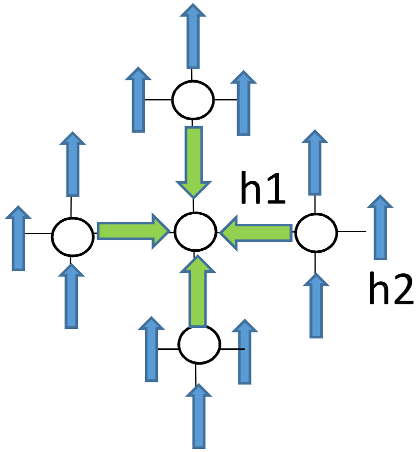
Mean field approximation

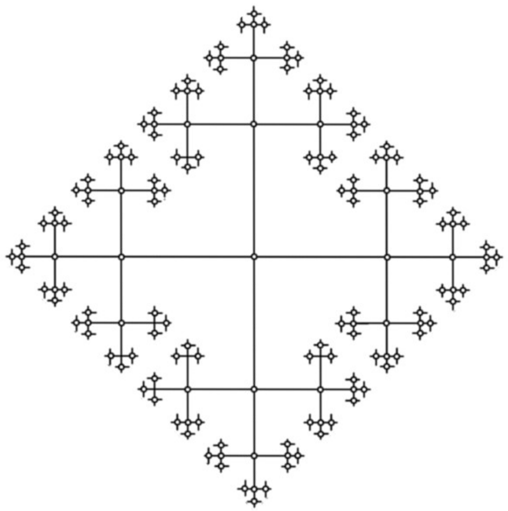




all to all interactions
(Husimi-Temperley model)

Bethe approximation





Bethe lattice model

Tensor network and holography

The MERA(1+1D) describes the area law of entanglement entropy up to log correction.

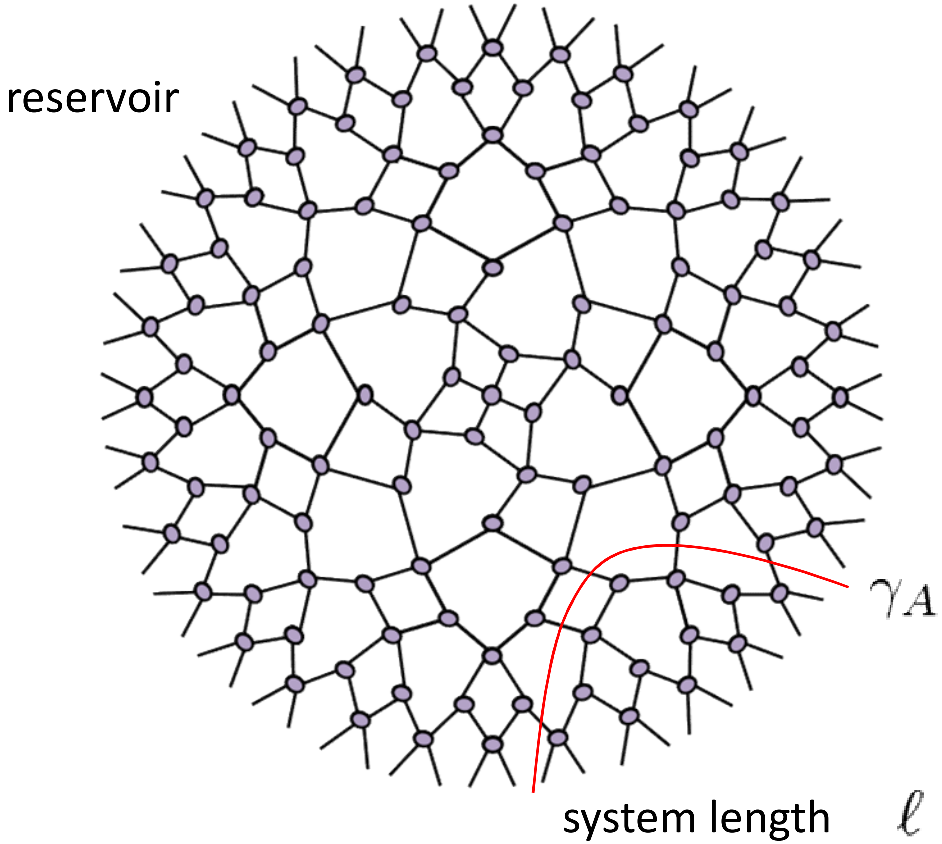
$\Rightarrow$ Ryu-Takayanagi formula (AdS side)

However

- tensor network states are usually variational wavefunction (CFT side)
- MERA tensor elements are highly nontrivial because of numerical optimization

Is there any simple tensor network enabling us to analytically describe the AdS side?

Tensor network---MERA/TNR



MERA network representation

Variational wavefunction (CFT side)

The EE between a system and the rest

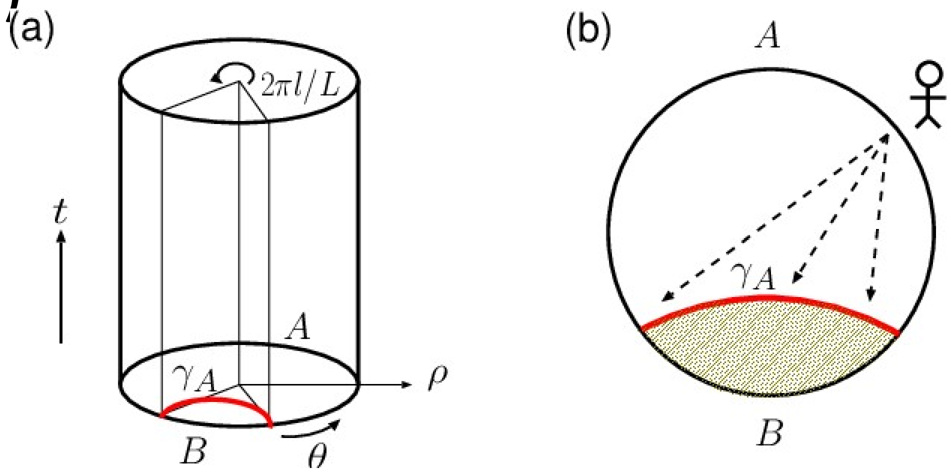
$$\gamma_A \sim \# \text{ of bonds}$$

$$S_{\mathrm{EE}} \sim \log \ell$$

- Holography (AdS/CFT)

$$S_{\mathrm{EE}} = \frac{\text{Area of}\,\gamma_A}{4G}$$

Ryu-Takayanagi formula



- Correlation function

$$\left\langle \exp\left(i \int_{\partial M} \phi_0 \mathcal{O} \right) \right\rangle = e^{i S_{\text{on shell}}[\phi_0]}$$

AdS/CFT Dictionary

CFT at the boundary

Bulk Gravity

types of TNs

Tree-type TN — Gapful systems

Practical TN methods!

MPS： DMRG・CTMRG

Tree： block spin transformations

TRG, HOTRG

(perfect binary tree)

Partial loops — 2D TPS/PEPS, MPS for the PBC

Loop TN — Critical systems; log-correction to EE

Tree ＋ loop networks — TNR（TRG＋loops）

MERA（block spin + loops）

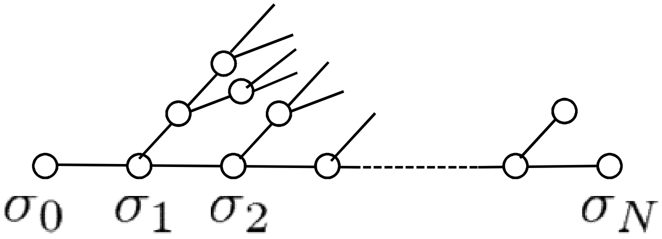
Cayley tree Ising model

T.P. Eggarter Phys. Rev. B 9 2989 (1974)
E. Mullar-Hartmann and J. Zitterz , PRL 33 893 (1974)
R. J. Baxter, textbook, chap. 4

A tree tensor network (loop free)

analytic calculation! (1D model)

The center spin = Bethe Approximation (Mean field)

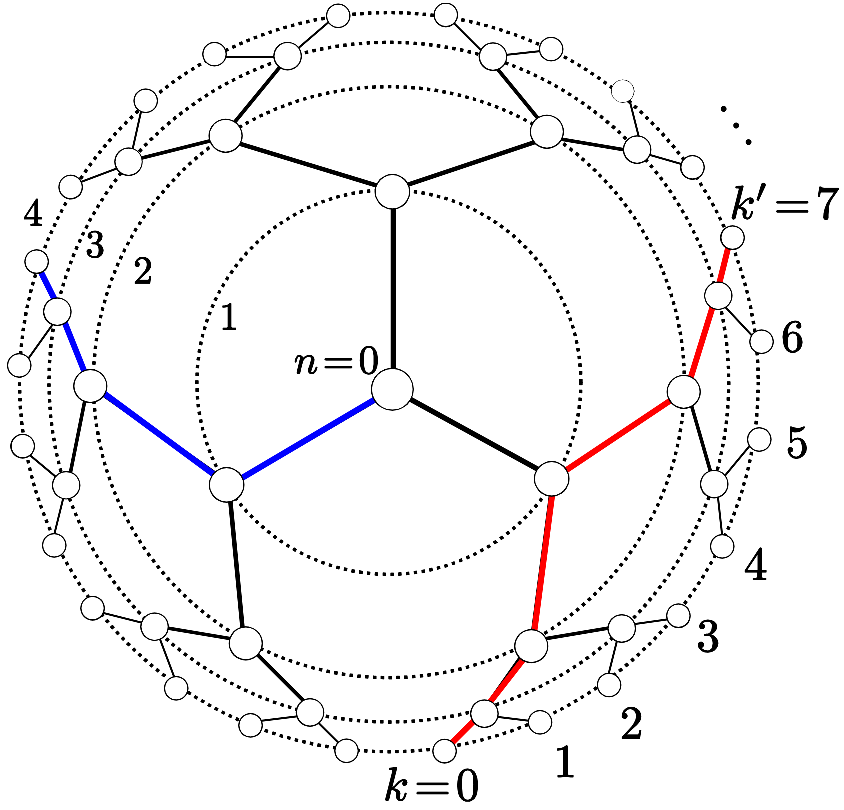


$$Z_N = \sum_{\sigma_0} e^{h\sigma_0}[g_N(\sigma_0)]^q$$

$$g_N(\sigma_0) = \sum_{\sigma_1} e^{K\sigma_0\sigma_1}[g_{N-1}(\sigma_1)]^p$$

$$= \sum_{\sigma_1} e^{K\sigma_0\sigma_1}\left[\sum_{\sigma_2} e^{K\sigma_1\sigma_2}[g_{N-2}(\sigma_2)]^p\right]^p$$

$$= \cdots$$



boundary spins $q(q-1)^{N-1}$

Coordination number q=3(Cayley tree)
p = q-1 : branching number

Recursion relation for the effective magnetic field

$$C_{n-1}e^{\tilde{h}_{n-1}\sigma_{n-1}} = \left[\sum_{\sigma_n} e^{K\sigma_{n-1}\sigma_n + \tilde{h}_n\sigma_n}\right]^p$$

$$\Rightarrow \quad \tilde{h}_{n-1} = f(\tilde{h}_n)$$

with $\quad f(x) \equiv \dfrac{p}{2}\log\left(\dfrac{e^{K+x}+e^{-K-x}}{e^{K-x}+e^{-K+x}}\right)$

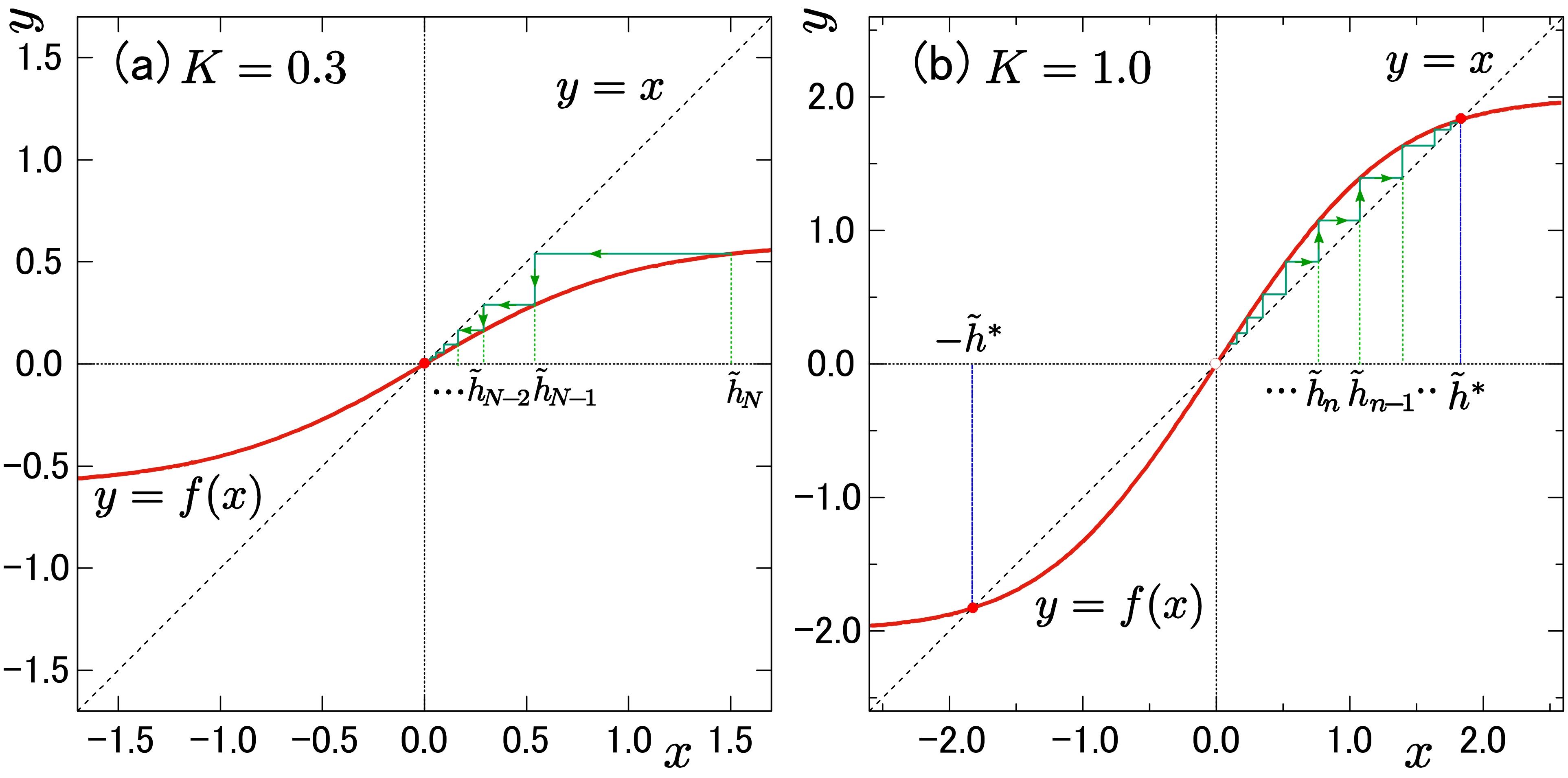
$\tilde{h}_N \equiv h_b$ boundary magnetic field (no magnetic field in the bulk)

This can be viewed as a holographic RG flow from the boundary to the Bulk

Bethe Approximation $\quad \tilde{h}^* = f(\tilde{h}^*)$

Self-consistent equation = fixed point of RG flows

Holographic RG flow from the boundary to the bulk



Disorder state (T< T_B) $T_{\mathrm{B}} = \frac{1}{2}\log\frac{q}{q-1}$ order state (T> T_B)

Effective magnetic field around the fixed point

$$\tilde{h}_n \simeq h_b \lambda^{N-n} \sim h_b \lambda^{-n} \quad \text{with} \quad \lambda = p \tanh K$$

Correlation function for boundary spins ($h_b = 0$)

$$\langle \sigma_{Nk} \sigma_{Nk'} \rangle = (\tanh K)^{2d_{kk'}}$$

$d_{kk'} = N - n_{kk'}$ specifies the lattice distance for two spins

For the redline path $N = 4,\ k = 0,\ k' = 7$

$\Rightarrow$ $n_{kk'} = 1$ and $d_{kk'} = 3$

The both exhibits the exponential decay at the level of the lattice network!

The boundary spin operator for $h_b = 0$

$$\sum_{\sigma_0 \cdots \sigma_N} \sigma_N e^{K \sum_{i=0}^{N-1} \sigma_i \sigma_{i+1}}$$

$$= 2\cosh K \sum_{\sigma_0 \cdots \sigma_{N-1}} [\tanh K \sigma_{N-1}] e^{K \sum_{i=0}^{N-2} \sigma_i \sigma_{i+1}}$$

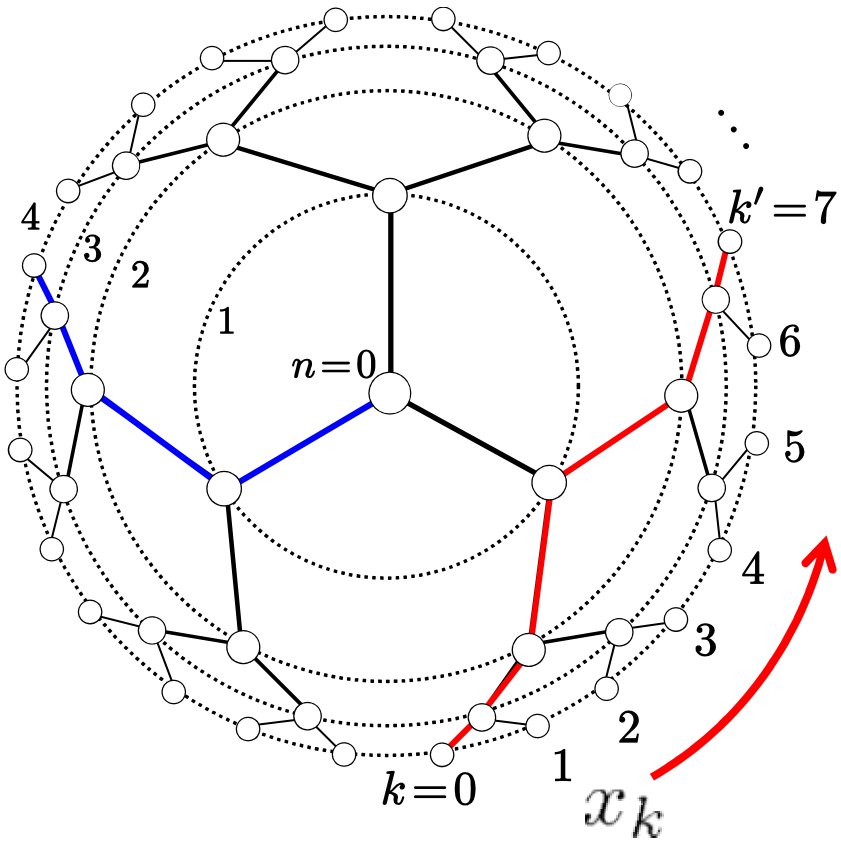
$$\cdots$$

$$= (2\cosh K)^{N-n} \sum_{\sigma_0 \cdots \sigma_n} [(\tanh K)^{N-n} \sigma_n] e^{K \sum_{i=0}^{N-n+1} \sigma_i \sigma_{i+1}}$$

$$\Rightarrow \quad (\tanh K)^{N-n} \sigma_n = \sigma_N$$

consistent with the boundary correlation function

network geometry



Unit disk

Radial direction

$$R_n = 1 - z_n$$

$$z_n = p^{-n}$$

Circumference direction

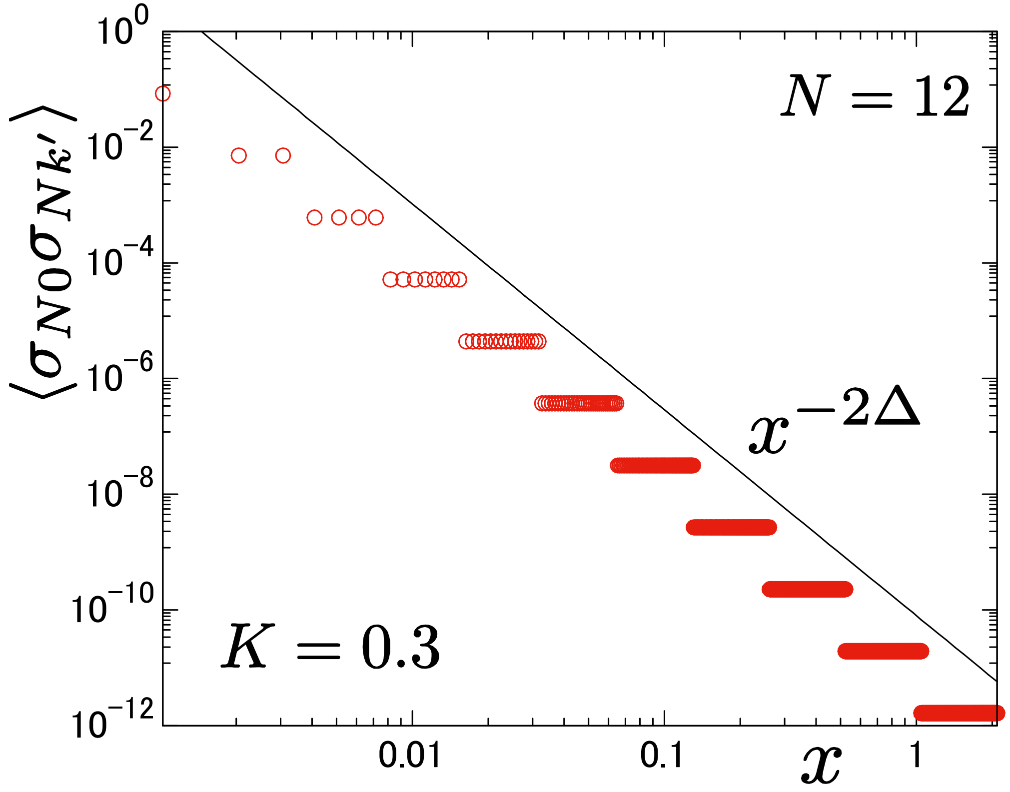
$$x_k = \frac{2\pi R_n}{q p^{n-1}} k \simeq \frac{2\pi p}{q} z_n k,$$

$$\text{for} \quad n \gg 1$$

$$\tilde{h}_n \sim \lambda^{-n} = z_n^{1-\Delta}$$

$$\langle \sigma_{N0} \sigma_{Nk'} \rangle \sim \frac{1}{(x_{k'} - x_0)^{2\Delta}}$$

$$\Delta \equiv -\frac{\log(\tanh K)}{\log p}$$


$\langle \sigma_{N0}\sigma_{Nk'} \rangle$
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0.01
0.1
1
x
$N = 12$
$x^{-2\Delta}$
$K = 0.3$

Unit disk representation

$$(z_n, x_k) \Rightarrow ds^2 = \frac{L^2}{z^2}(dz^2 + d\tilde{x}^2)$$

near the boundary

$$L \equiv \frac{1}{\log p}$$

A scalar field in AdS

$$\phi(z) \sim Az^{\Delta_-} + Bz^{\Delta_+} \quad \text{with} \quad \Delta_+ + \Delta_- = d$$

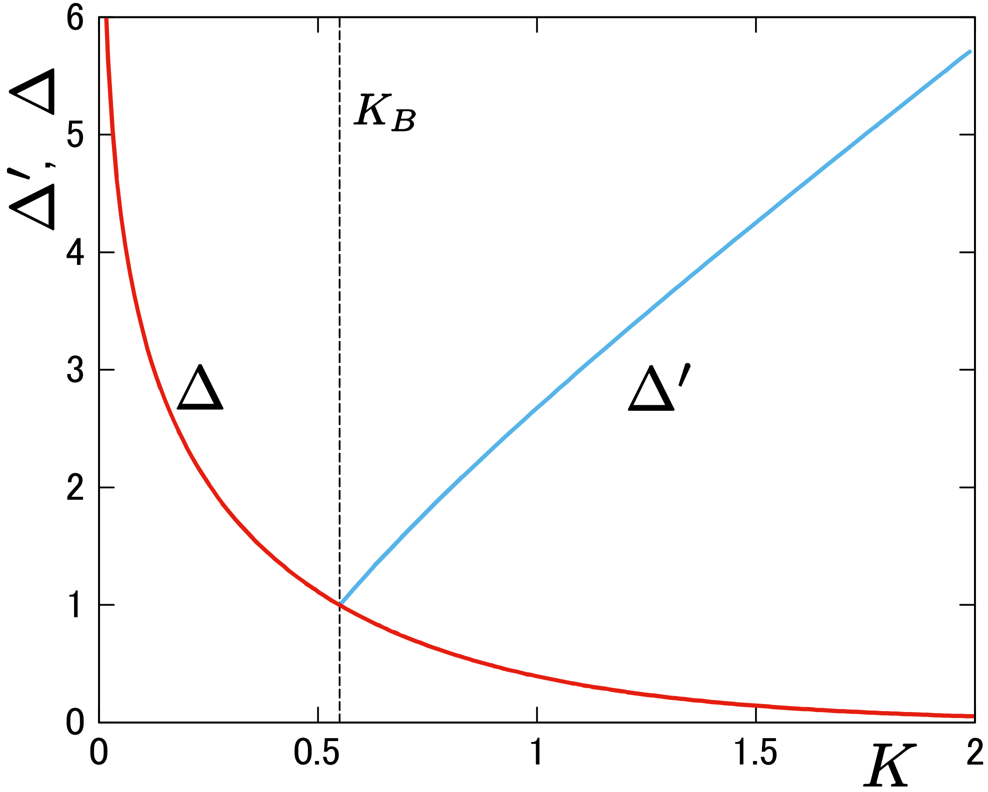
$$\boxed{\begin{array}{l} A \leftrightarrow h_b \\ B \leftrightarrow \sigma_N \quad \text{with} \quad d = 1 \\ \Delta_+ \leftrightarrow \Delta \end{array}}$$

$$\Delta \equiv -\frac{\log(\tanh K)}{\log p}$$

Notes:

*The correlation path can be described by a geodesic in Poincare coordinate

* EE for the Bethe lattice is bounded because of the tree network nature.



Bethe transition point corresponds to Δ=1 (massless point)

$$\Delta \equiv -\frac{\log(\tanh K)}{\log p}$$

$$\Delta' = -\frac{\log\left(\frac{\sinh(2K)}{\cosh(2K)+\cosh(2\tilde{h}_*)}\right)}{\log p}$$

T< T_B

Relation to the p-adic AdS(Burhat-Tit Tree)

p=2 (Cayley tree)

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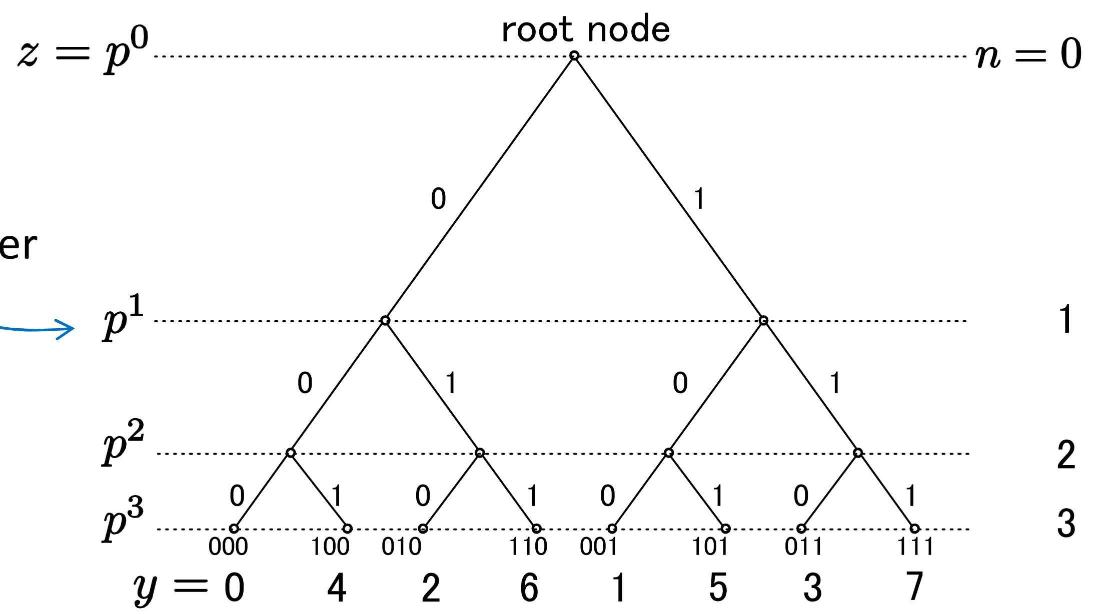
Digit for the prime number



p-adic number $y = \sum c_n p^n$, where $c_n = 0, 1, \cdots p-1$

p-adic norm: $|y|_p = p^{-m}$ if $y = a/b \cdot p^m$

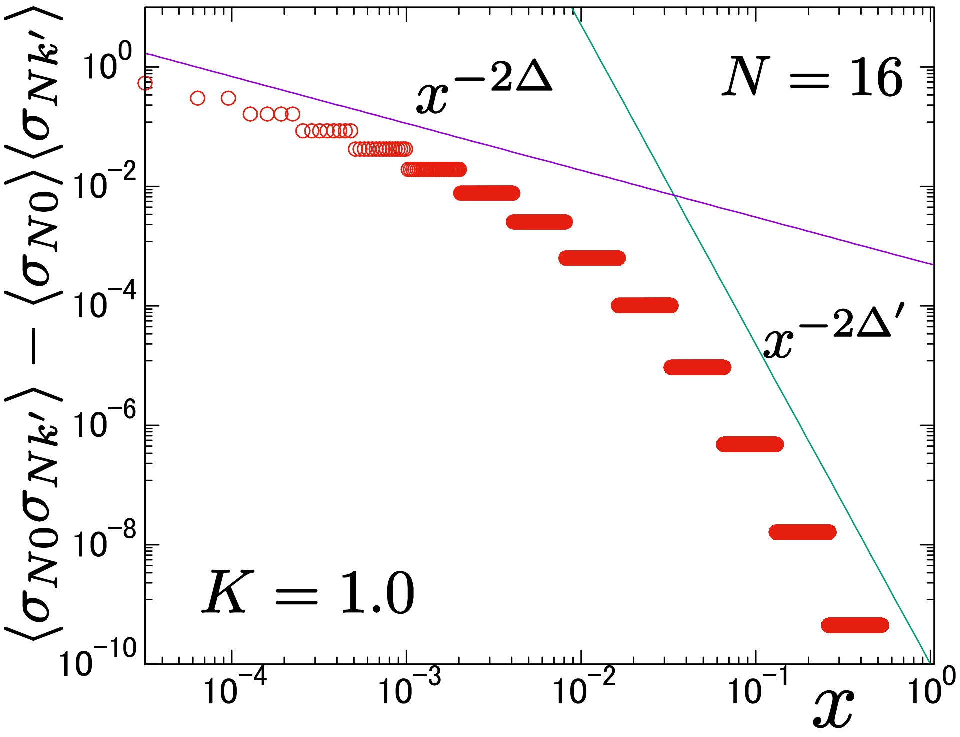
$$\phi_p(z) \sim |z|_p^{d-\Delta} \sim p^{-(d-\Delta)n}$$

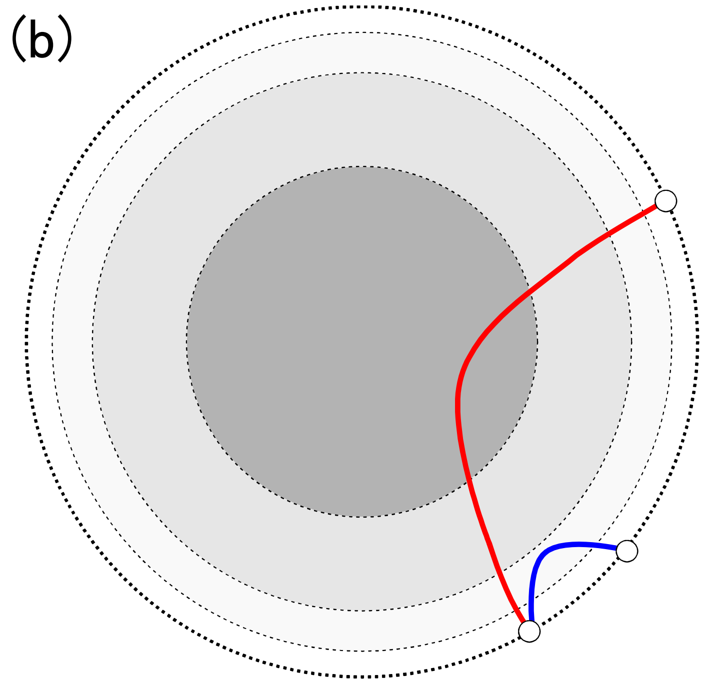
$$\langle \phi_p(y)\phi_p(y') \rangle \sim \frac{1}{|y-y'|_p^{2\Delta}}$$

$$\tilde{h}_n \sim p^{-(1-\Delta)n}$$

$$\langle \sigma_y \sigma_{y'} \rangle \sim \frac{1}{|y-y'|_p^{2\Delta}}$$

Ferromagnetic fixed point and crossover for $T < T_B$





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