The nonperturbative behavior of the tricritical and tetracritical fixed points of the $O(N)$ models at large N

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O(N) models

- They have played an important role in our understanding of second order phase transitions.
- N-component vector order parameter
N=1…Ising, N=2…XY, N=3…Heisenberg Model
- Critical physics (controled by WF FP) has been well understood with various theoretical approaches…Exact solution (2d Ising), Renormalization group（d=4-ε, 2+ε expansion), Large-N analysis, conformal bootstrap

We study multicrical fixed points at Large-N and show that nonperturbative effects are important.

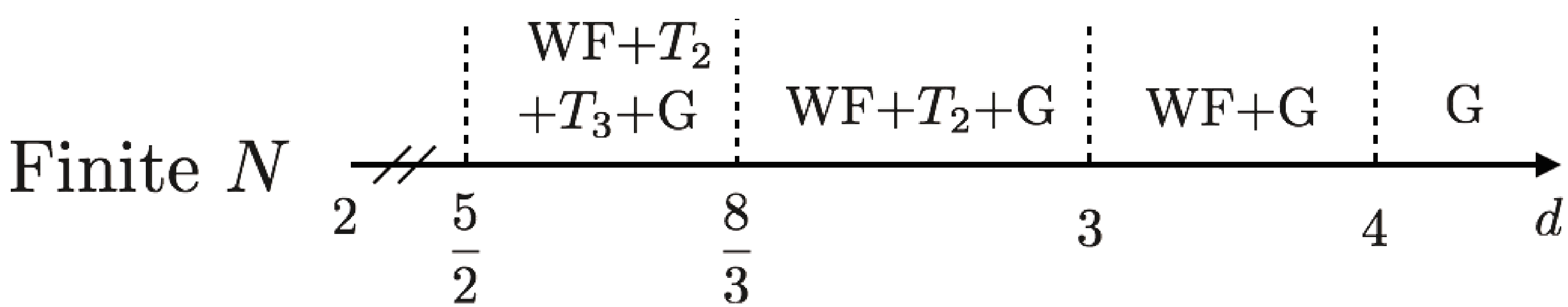
Common wisdom on the criticality of O(N) models (finite N case)

GLW Hamiltonian $H[\phi] = \frac{1}{2}\int_x (\nabla\phi_i)^2 + U(\phi)$ ϕ_i

$U(\phi) = a_2\phi_i^2 + a_4(\phi_i^2)^2 + a_6(\phi_i^2)^3 + \dots$

N-component order parameter

Below the critical dimension $d_n = 2 + 2/n$, the $(\phi_i^2)^{n+1}$ term becomes relevant around the Gaussian FP (G).



A nontrivial fixed point T_n with n relevant (unstable) directions branches from G at d_n. (Wilson-Fisher FP, which describes second order phase transition, at d=4 and the tricritical FP T_2 at d=3….)

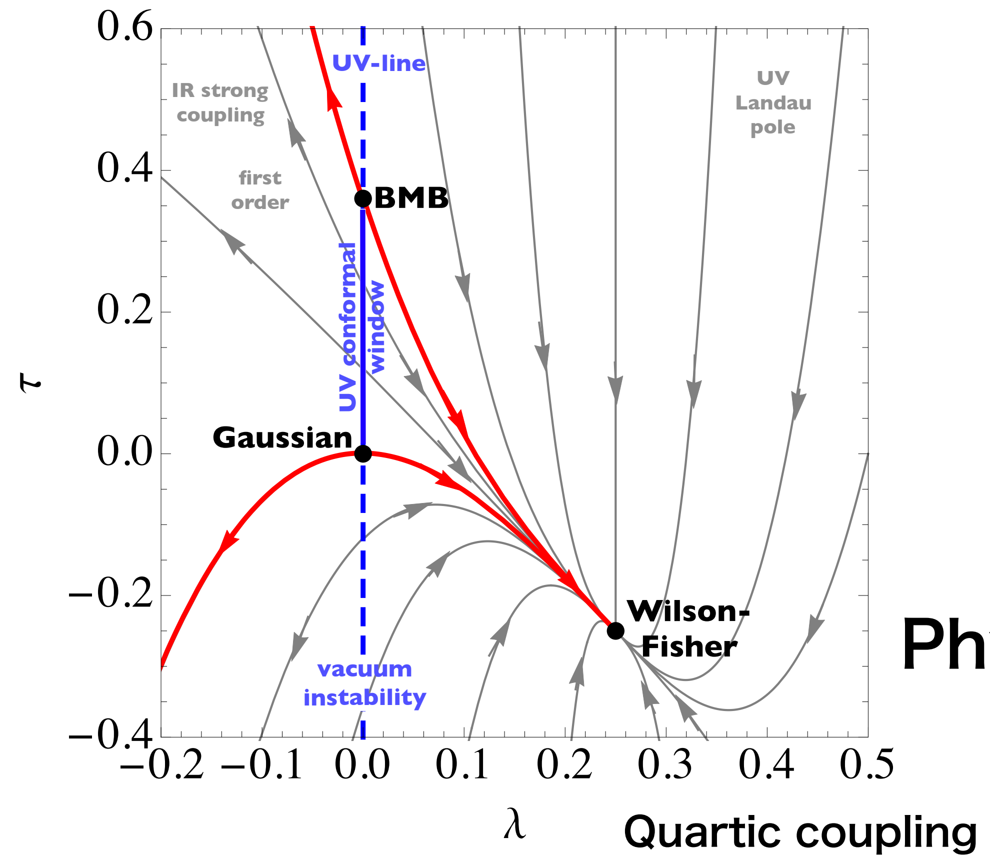
Bardeen-Moshe-Bander (BMB) Phenomena at $d = 3$ and $N = \infty$

Bardeen-Moshe-Bander found an intriguing phenomenon in $O(N)$ models with $\tau(\varphi^2)^3$ interaction.

(i) at $d = 3$ and $N = \infty$, there exists a finite line of fixed points that starts at the Gaussian FP ($\tau = 0$) and ends at a special FP called the BMB FP

(ii) the FPs for $\tau \neq 0$ are interacting FP (2-unstable and 1-magrinal) but the critical exponents are all identical to the Gaussian ones.

Bardeen-Moshe-Bander (BMB) Phenomenon at $d = 3$ and $N = \infty$



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(iii) The FPs are UV stable on the critical surface

(iv) The FP potential of BMB FP has singularity at small fields, suggesting spontaneous symmetry breaking of scale invariance

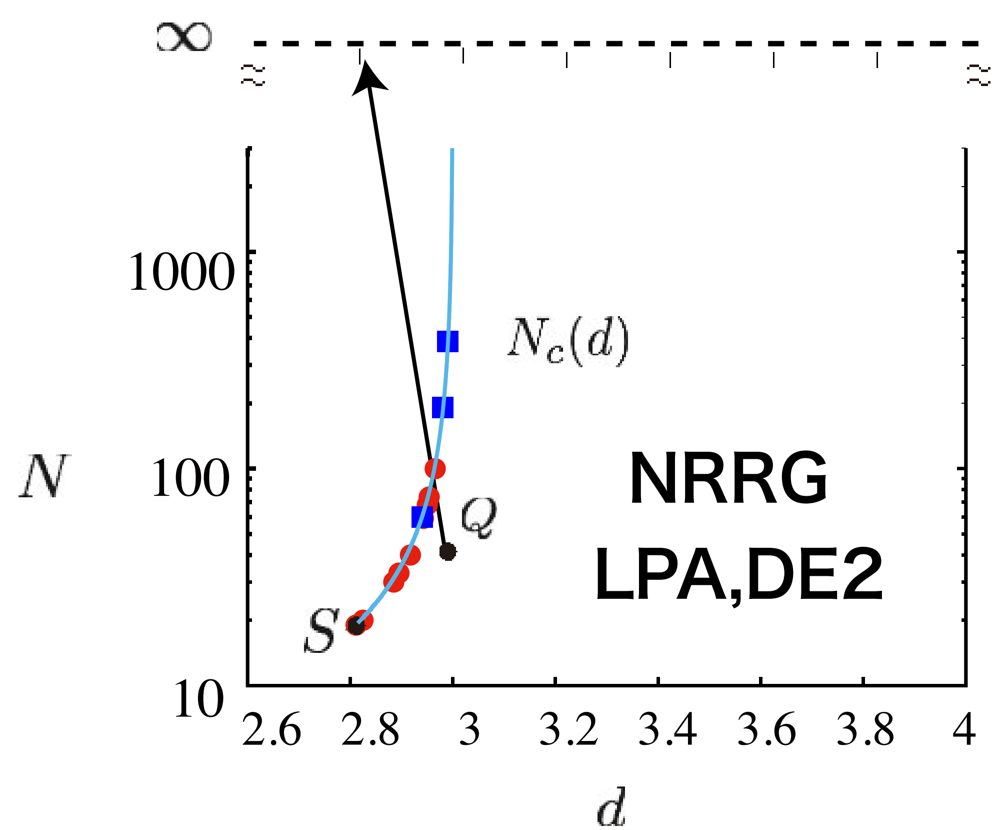
Flow of τ

- The flow of $\tilde{\tau} = N^2\tau$ in perturbation theory at Large N becomes
  $N\beta_{\tilde{\tau}} = -2\alpha\tilde{\lambda} + 12\tilde{\lambda}^2 - \pi^2\tilde{\lambda}^3/2 + O(1/N)$
  where $\alpha = (3-d)N$

- Two roots of $\beta_{\tilde{\tau}} = 0$: $\tilde{\tau}_{\pm} = 4/\pi^2(3 \pm \sqrt{9 - \pi^2\alpha/4})$
  $\tilde{\tau}_{-}$ (corresponding to the perturbative tricritical FP) exists
  between $3 - \alpha_c/N < d < 3$ with $\alpha_c = 36/\pi^2 \simeq 3.65$.
  $\tilde{\tau}_{+}$ exists for $3 - \alpha_c/N < d$.

  (Note that perturbation theory might be not very precise for $\tilde{\tau}_{+}$)

A_2 , $\tilde{A}_3$



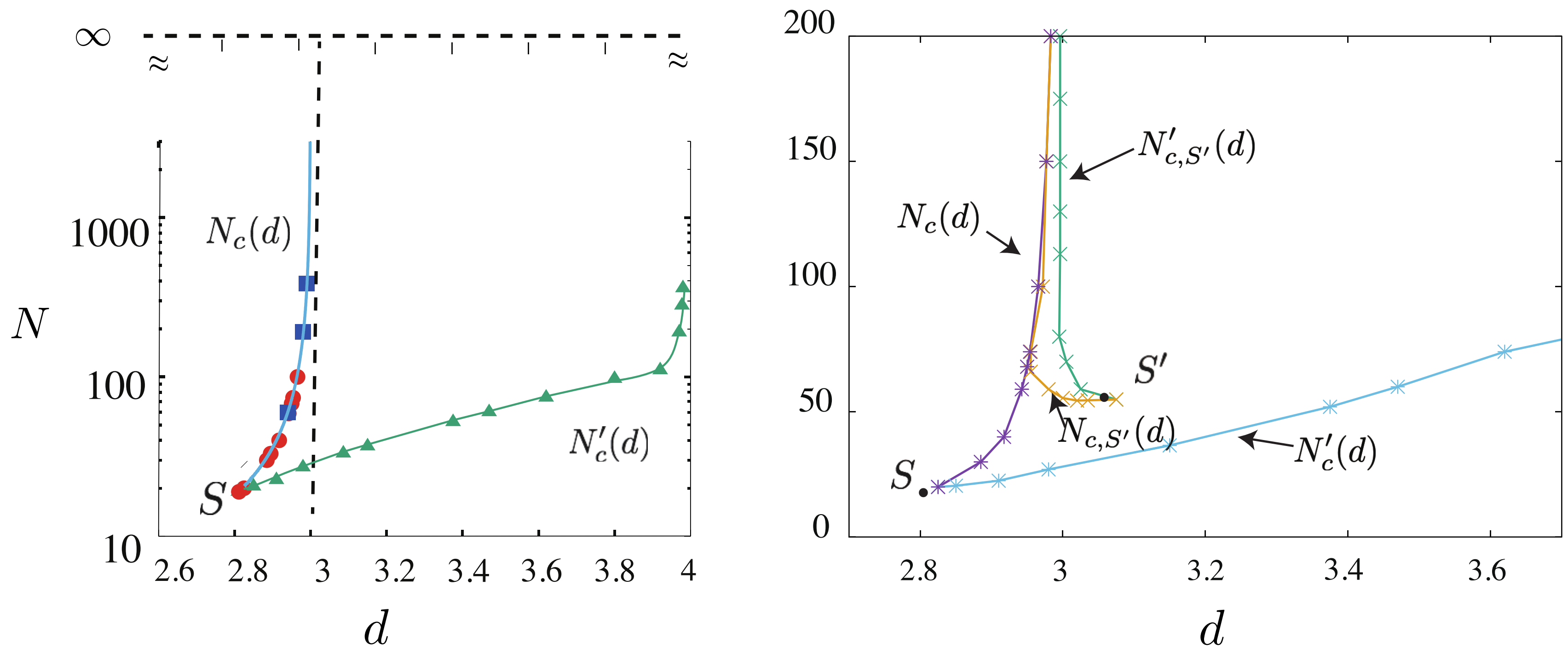
- Hereafter we call the FP corresponding $\tilde{\tau}_-$ ($\tilde{\tau}_+$) as A_2 ($\tilde{A}_3$)

  The number of relevant directions around a FP is indicated with the subscript.

- How can we understand the line of FPs and its end point at $N = \infty$ using

  Large-N limit of finite N FPs (A_2 , $\tilde{A}_3$)?

Fixed point structure at general values of (d, N)



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- Understanding BMB phenomena is essential for understanding fixed point structure of $O(N)$ models at general values of (d, N).

Bardeen-Moshe-Bander (BMB) line at $d = 8/3$ and $N = \infty$

It is shown with FRG that
(J. Comellas and A. Travesset, Nucl. Phys. B 1997),

(i) at $d = 8/3$ and $N = \infty$, there exists a finite line of tetracritical fixed points (3 unstable+1 marginal) that starts at the Gaussian FP ($\lambda = 0$) and ends at the WF FP.
(Question: How can the stabilities of the FPs be different?)

(ii) the FPs ($\neq$ WF) for $\lambda \neq 0$ are interacting FP but the critical exponents are all identical to the Gaussian ones.

Flow of λ

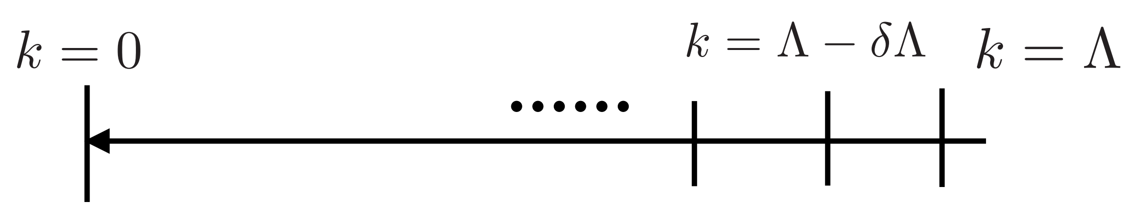
- Calling $\lambda/(384N^3)$ the coupling in front of $\lambda(\varphi^2)^4$ term, the large-N flow equation for λ becomes

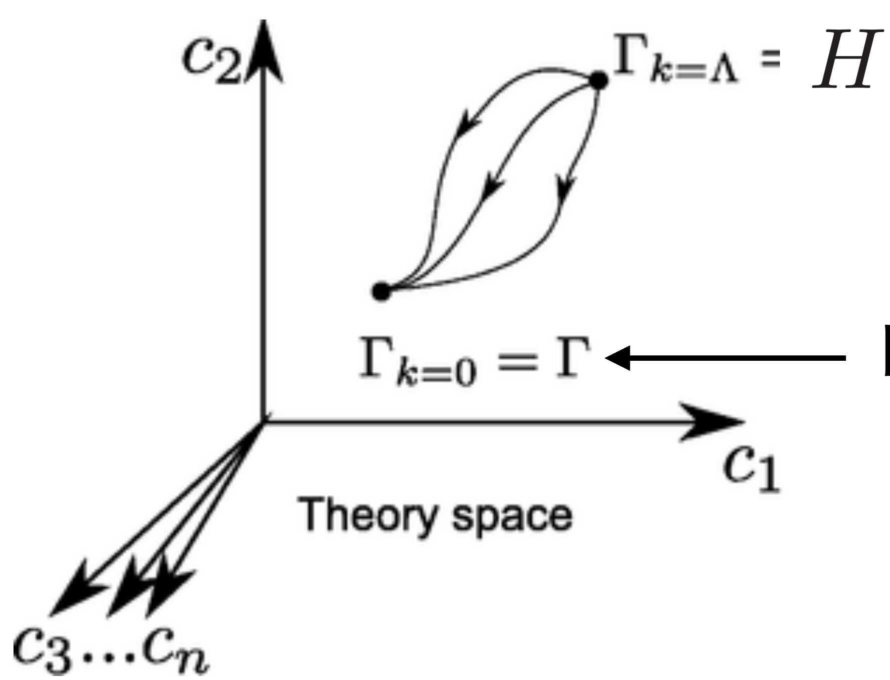
  $\partial_t \lambda = -3\epsilon\lambda + \dfrac{9\lambda^2}{4N} + O(N^{-2})$ with $\epsilon = 8/3 - d$

- The nontrivial tetracritical FP solution is $\lambda* = 4\epsilon N/3$ and does not indicate disappearance of FP in the limit $N \to \infty$ at fixed ϵ. (However, in this limit, the FP is no longer controlled perturbatively.)

Non perturbative renormalization group (NPRG)

- Modern implementation of Wilson's RG that takes the fluctuation into account step by step in lowering the cut-off wavenumber k, in terms of wavenumber-dependent effective action Γ_k





Microscopic Hamiltonian

Effective action (Free energy), where all the fluctuations are taken into account.

NPRG equation

NPRG equation (Wetterich, Phys. Lett. B, 1993) is

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}[\partial_t R_k(q^2)(\Gamma_k^{(2)}[q,-q;\phi] + R_k(q))^{-1}]$$

$$t = \ln{(k/\Lambda)}$$

Derivative expansion(DE2)

- It is impossible to solve the NPRG equation exactly and we have recourse to approximations,

$$\Gamma_k[\phi] = \int_x \left(\frac{1}{2} Z_k(\rho)(\nabla\phi_i)^2 + \frac{1}{4} Y_k(\rho)(\phi_i \nabla \phi_i)^2 + U_k(\rho) + O(\nabla^4) \right).$$

$$\rho = \phi_i \phi_i / 2$$

- Simpler approximations…LPA(η=0), LPA' approximation

$$Y_k(\rho) = 0$$

$$Z_k(\rho) = \bar{Z}_k$$

$$\downarrow$$

$$\eta_t = -\partial_t \log \bar{Z}_k$$

Applications of DE

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Reexamination of the nonperturbative renormalization-group approach to the Kosterlitz-Thouless transition

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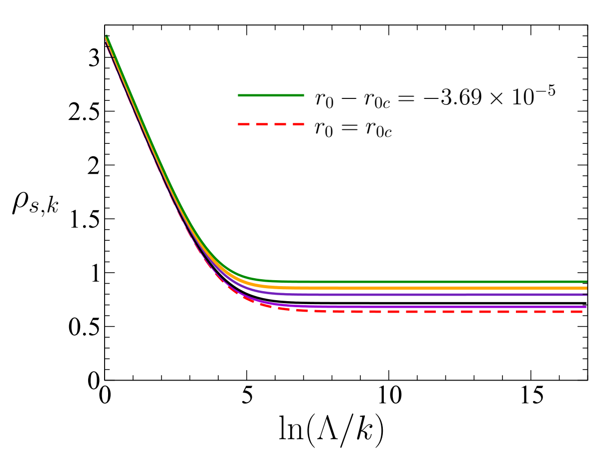
We reexamine the two-dimensional linear O(2) model (φ^4 theory) in the framework of the nonperturbative renormalization-group. From the flow equations obtained in the derivative expansion to second order and with optimization of the infrared regulator, we find a transition between a high-temperature (disordered) phase and a low-temperature phase displaying a line of fixed points and algebraic order. We obtain a picture in agreement with the standard theory of the Kosterlitz-Thouless (KT) transition and reproduce the universal features of the transition. In particular, we find the anomalous dimension $\eta(T_{\rm KT}) \simeq 0.24$ and the stiffness jump $\rho_s(T_{\rm KT}^-) \simeq 0.64$ at the transition temperature $T_{\rm KT}$, in very good agreement with the exact results $\eta(T_{\rm KT}) = 1/4$ and $\rho_s(T_{\rm KT}^-) = 2/\pi$, as well as an essential singularity of the correlation length in the high-temperature phase as $T \to T_{\rm KT}$.

$$\Delta\Gamma_k[\boldsymbol{\phi}] = \frac{1}{2}\rho_{s,k}\int d^d r\,(\nabla\theta)^2.$$



Precision calculation of critical exponents in the O(N) universality classes with the nonperturbative renormalization group

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We compute the critical exponents ν, η and ω of $O(N)$ models for various values of N by implementing the derivative expansion of the nonperturbative renormalization group up to next-to-next-to-leading order [usually denoted $O(\partial^4)$]. We analyze the behavior of this approximation scheme at successive orders and observe an apparent convergence with a small parameter, typically between $\frac{1}{9}$ and $\frac{1}{4}$, compatible with previous studies in the Ising case. This allows us to give well-grounded error bars. We obtain a determination of critical exponents with a precision which is similar or better than those obtained by most field-theoretical techniques. We also reach a better precision than Monte Carlo simulations in some physically relevant situations. In the O(2) case, where there is a long-standing controversy between Monte Carlo estimates and experiments for the specific heat exponent α, our results are compatible with those of Monte Carlo but clearly exclude experimental values.

	ν	η	ω
LPA	0.7090	0	0.672
$O(\partial^2)$	0.6725(52)	0.0410(59)	0.798(34)
$O(\partial^4)$	0.6716(6)	0.0380(13)	0.791(8)
CB (2016)	0.6719(12)	0.0385(7)	0.811(19)
CB (2019)	0.6718(1)	0.03818(4)	0.794(8)
Six-loop, $d = 3$	0.6703(15)	0.0354(25)	0.789(11)
ϵ expansion, ϵ^5	0.6680(35)	0.0380(50)	0.802(18)
ϵ expansion, ϵ^6	0.6690(10)	0.0380(6)	0.804(3)
MC+High T (2006)	0.6717(1)	0.0381(2)	0.785(20)
MC (2019)	0.67169(7)	0.03810(8)	0.789(4)
Helium-4 (2003)	0.6709(1)		
Helium-4 (1984)	0.6717(4)		
XY-AF ($CsMnF_3$)	0.6710(7)		
XY-AF ($SmMnO_3$)	0.6710(3)		
XY-F (Gd_2IFe_2)	0.671(24)	0.034(47)	
XY-F (Gd_2ICo_2)	0.668(24)	0.032(47)	

Nondimensionalized NPRG eq.

- Scaling solutions can be found as FPs solution of **nondimensionalized** NPRG eq.

$$\tilde{\phi} = \sqrt{Z_k} k^{\frac{2-d}{2}} \phi \qquad \tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{U}_t(\tilde{\rho}) = k^{-d} U_k(\rho)$$

Litim cutoff $\quad y = \dfrac{q^2}{k^2} \qquad R_k(q^2) = Z_k k^2 y r(y) \qquad r(y) = (1/y - 1)\theta(1-y)$

Under LPA,

$$\partial_t \tilde{U}_t(\tilde{\phi}) = -d\, \tilde{U}_t(\tilde{\phi}) + \frac{1}{2}(d-2)\tilde{\phi}\, \tilde{U}_t'(\tilde{\phi}) + (N-1)\, \frac{\tilde{\phi}}{\tilde{\phi} + \tilde{U}_t'(\tilde{\phi})} + \frac{1}{1 + \tilde{U}_t''(\tilde{\phi})}.$$

Usual large N limit of the LPA flow

Rescaled finite N equation $\tilde{U}_t = N\bar{U}_t$ $\tilde{\phi} = \sqrt{N}\bar{\phi}$

$$\partial_t \bar{U}_t(\bar{\phi}) = -d\,\bar{U}_t(\bar{\phi}) + \frac{1}{2}(d-2)\bar{\phi}\,\bar{U}_t'(\bar{\phi}) + \left(1 - \frac{1}{N}\right)\frac{\bar{\phi}}{\bar{\phi} + \bar{U}_t'(\bar{\phi})} + \frac{1}{N}\frac{1}{1+\bar{U}_t''(\bar{\phi})}$$

- The terms proportional to 1/N are assumed to be subleading.
- At N=∞, the resulting NPRG eq without an explicit 1/N dependence was believed to be exact and can be solved exactly.

Wilson-Polchinski version of NPRG

Transformation of the variables

$$(U,\phi) \longleftrightarrow (V,\Phi)$$

$$V(\mu) = U(\phi) + (\phi - \Phi)^2/2$$

$$\phi - \Phi = -2\Phi V'(\mu)$$

$$\mu = \Phi^2$$

Rescaling in N $\bar{\mu} = \mu/N,\ \bar{V} = V/N$

LPA FP eq.

$$0 = 1 - d\,\bar{V} + (d-2)\bar{\mu}\bar{V}' + 4\bar{\mu}\bar{V}'^2 - 2\bar{V}' - \underline{\frac{4}{N}\bar{\mu}\,\bar{V}''}.$$

$1/N$ A small parameter

$\bar{V}''$ The highest order derivative

We have to deal with singular perturbation in general.

Usual large-N limit in the Wilson-Polchinski parametrization

$$0 = 1 - d\,\bar{V} + (d-2)\bar{\varrho}\bar{V}' + 2\bar{\varrho}\bar{V}'^2 - \bar{V}' - \cancel{\frac{2}{N}\bar{\varrho}\,\bar{V}''}$$

- In generic dimensions $2 < d < 4$, it has three solutions: Gaussian FP (G), Wilson Fisher FP (WF) and linear FP $\bar{V}(\bar{\varrho}) = \bar{\varrho}$ (discontinuity FP).

- In dimensions $d = 2 + 2/p$ with odd integer $p > 0$, $(\boldsymbol{\varphi}^2)^{p+1}$ term is marginal around G and a line of FPs starting from G and terminating at BMB FP appears.

Results on the BMB line at $d = 3$ and $N = \infty$

Tricritical FP solutions in $d = 3$ and at $N = \infty$ in LPA

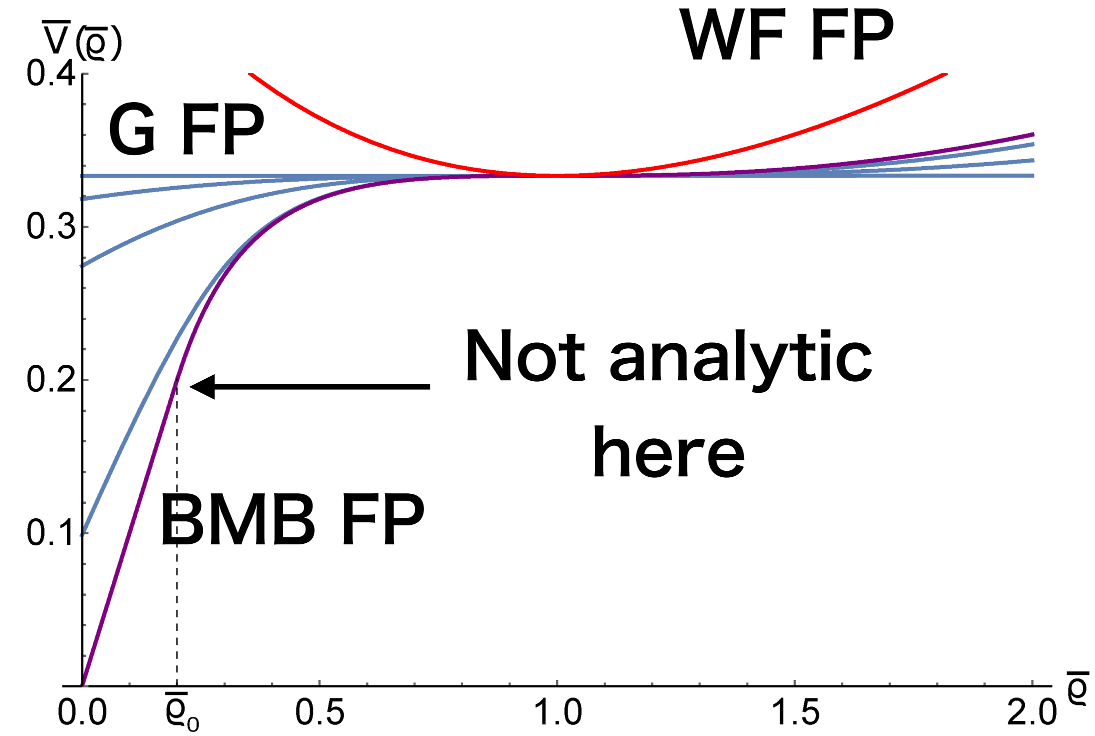
$$\bar{\varrho} = \bar{\mu}/2$$

$$\bar{\varrho}_{\pm} = 1 + \frac{\bar{V}'\left(\frac{5}{2} - \bar{V}'\right)}{\left(1 - \bar{V}'\right)^2} + \frac{\frac{3}{2}\arcsin\sqrt{\bar{V}'} \pm \sqrt{2/\tau}}{\left(\bar{V}'\right)^{-1/2}\left(1 - \bar{V}'\right)^{5/2}}$$

$\bar{\varrho}_{+} \dashrightarrow \bar{\varrho} > 1$

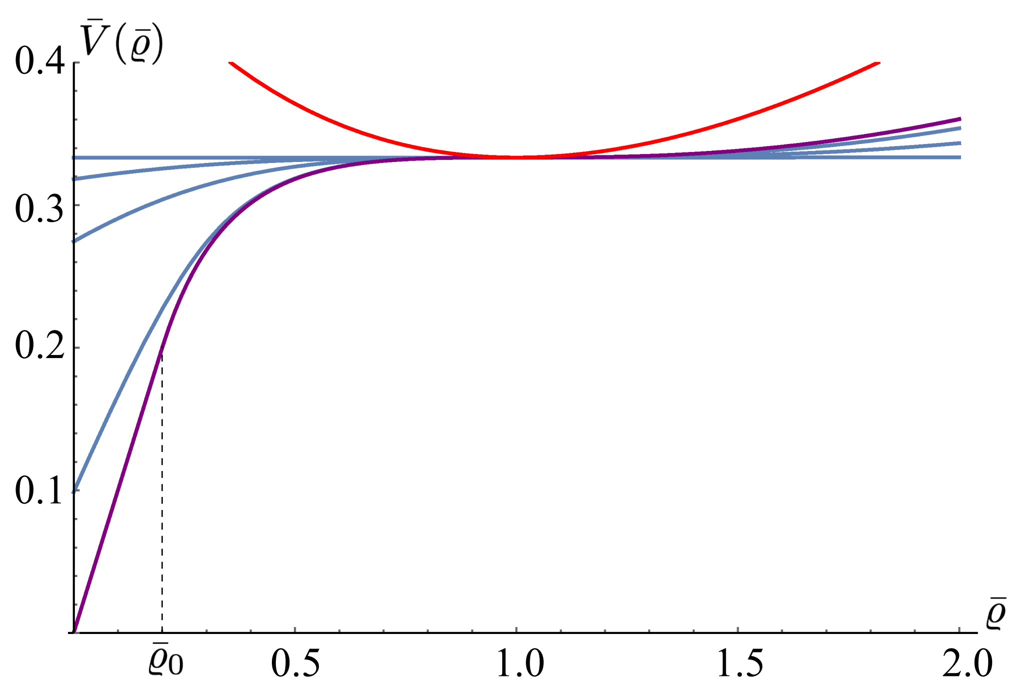
$\bar{\varrho}_{-} \dashrightarrow \bar{\varrho} < 1$

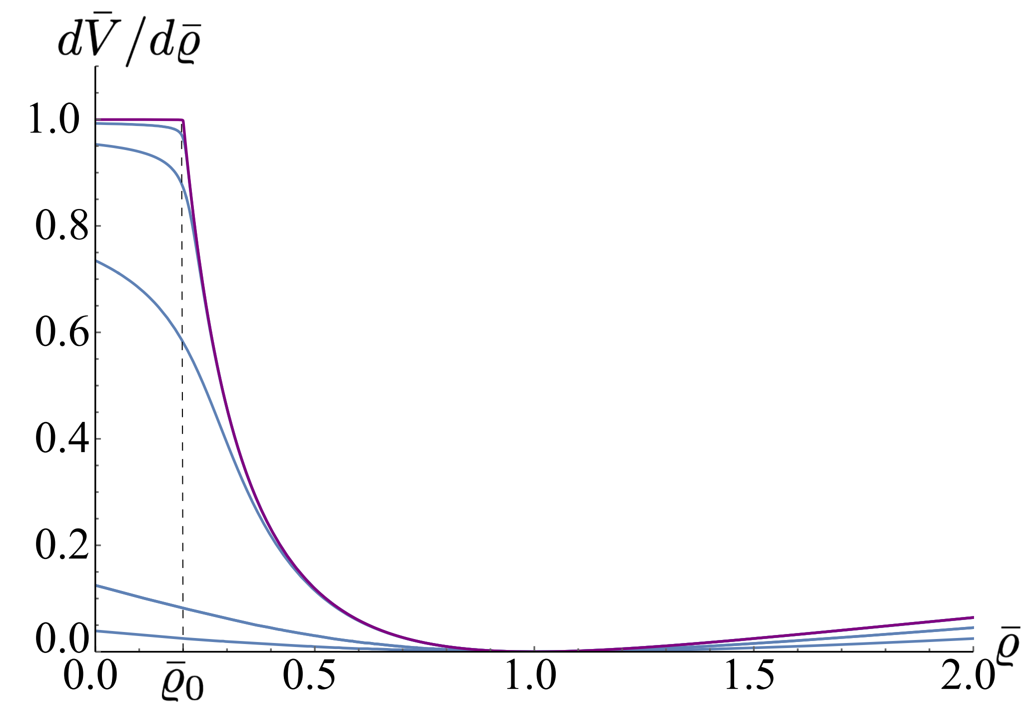
D. F. Litim and M. J. Trott, PRD (2018)



- $\tau = 0$ $\cdots$Gaussian (G) FP
- $\tau \in [0, \tau_{\rm BMB} = 32/(3\pi)^2]$ $\cdots$ FPs on the BMB line
- $\tau > \tau_{\rm BMB}$ $\cdots$No FP defined for all ϱ
- $\sqrt{2/\tau} = 0$ $\cdots$Wilson-Fisher (WF) FP

Nonanalicity of BMB FP





- $d^2\bar{V}/d\bar{\rho}^2$ becomes discontinuous at $\bar{\varrho} = \bar{\varrho}_0$.
- In Wetterich parametrization, the nonanalicity corresponds to a cusp-like behavior $\bar{U}(\bar{\phi}) \sim \mathrm{const}|\phi|$

Finite-N realization of the regular BMB line

- Let us consider to follow A_2 or $\tilde{A}_3$ on a path toward $(d=3, N=\infty)$: $d = 3 - \alpha/N\,,\, N \to \infty$

- It approaches a FP on the BMB line and τ is given by

$$\alpha - 36\tau + 96\tau^2 = 0$$

- Derivation: We expand the potential as

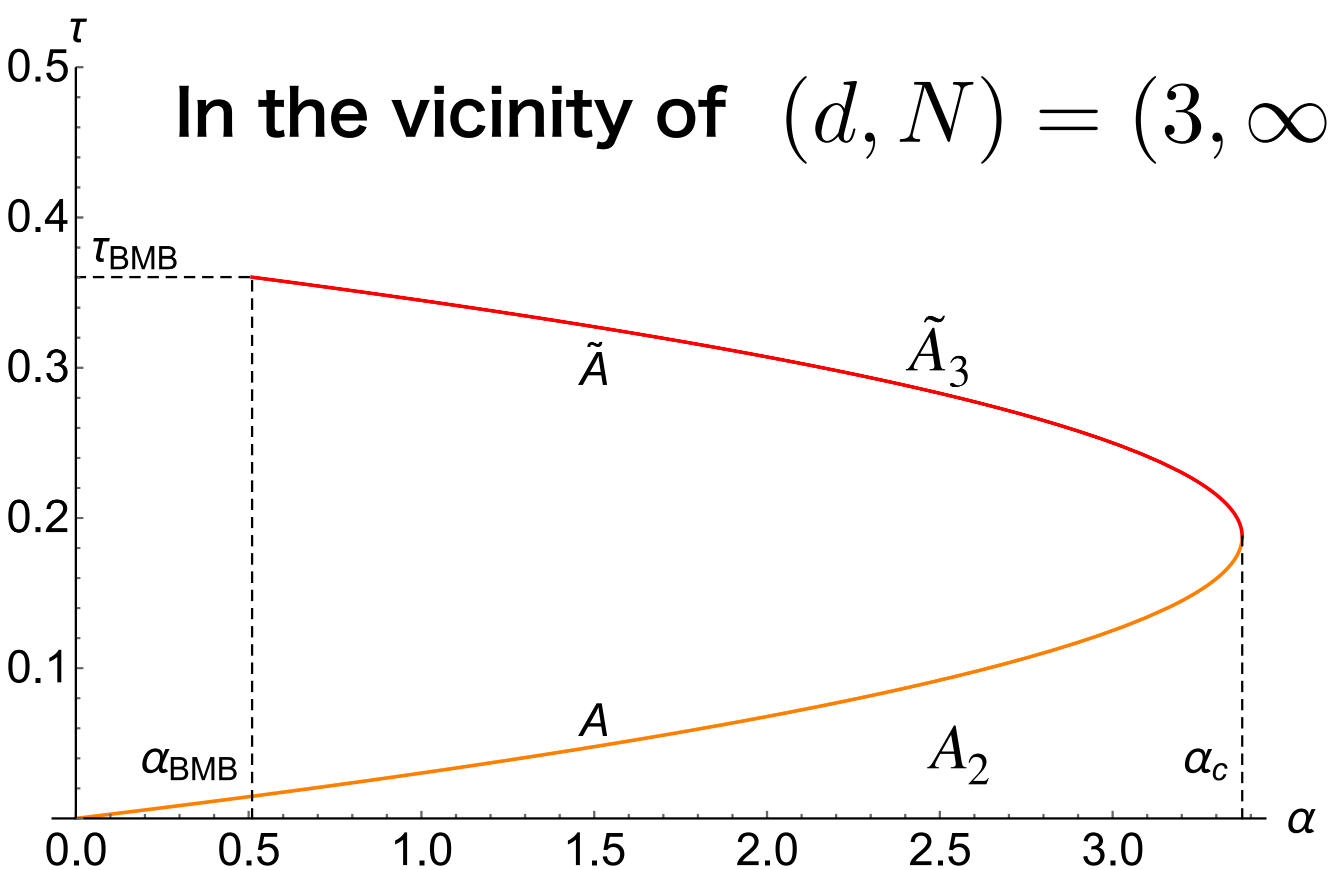
$$\bar{V}_{\alpha,N}(\bar{\varrho}) = \bar{V}_{\alpha,N=\infty}(\bar{\varrho}) + \bar{V}_{1,\alpha}(\bar{\varrho})/N + O(1/N^2).$$

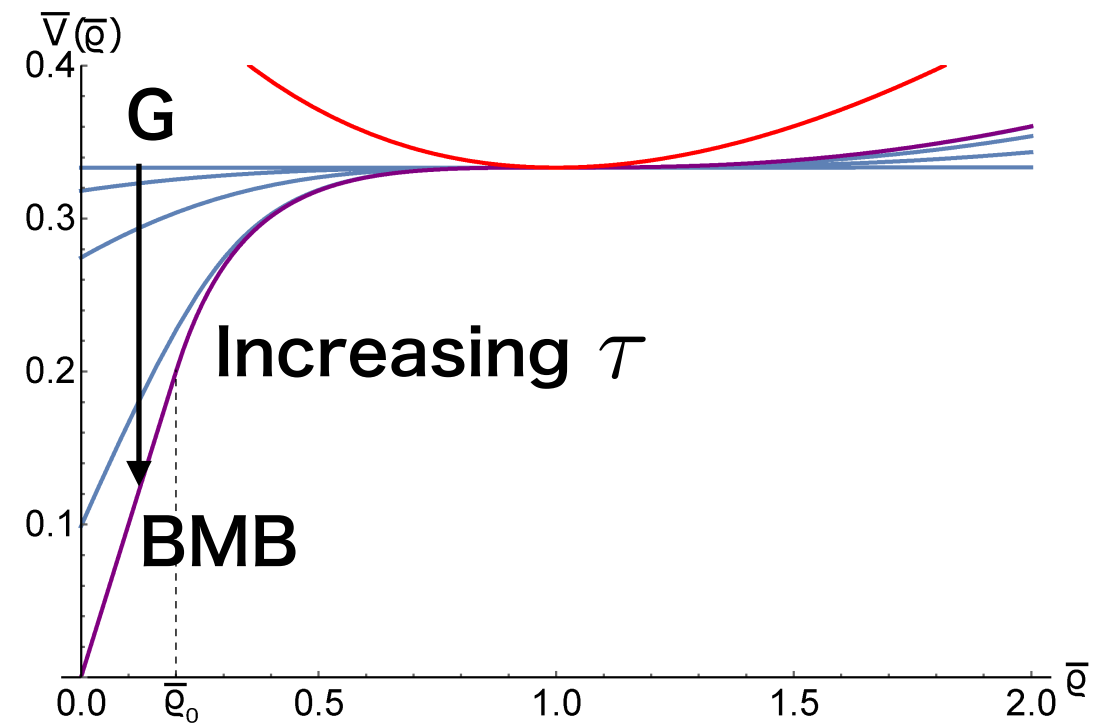
and impose analyticity of $\bar{V}_{1,\alpha}(\bar{\varrho})$ around $\bar{\varrho} = 1$

Plot of τ as a function of α

Under the double limit $d = 3 - \alpha/N, N \to \infty$

In the vicinity of $(d, N) = (3, \infty)$





- What occurs for $\tilde{A}_3$ at $\tau = \tau_{\mathrm{BMB}}$ and finite but large N?

Singular FPs constructed from the FPs on the BMB line

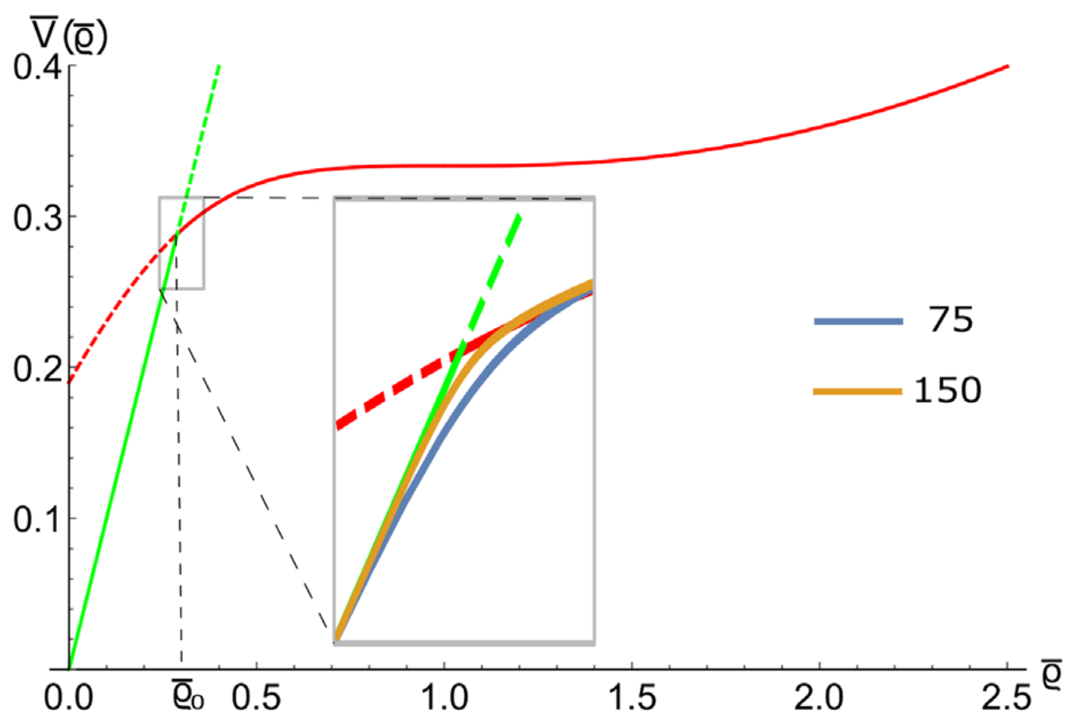
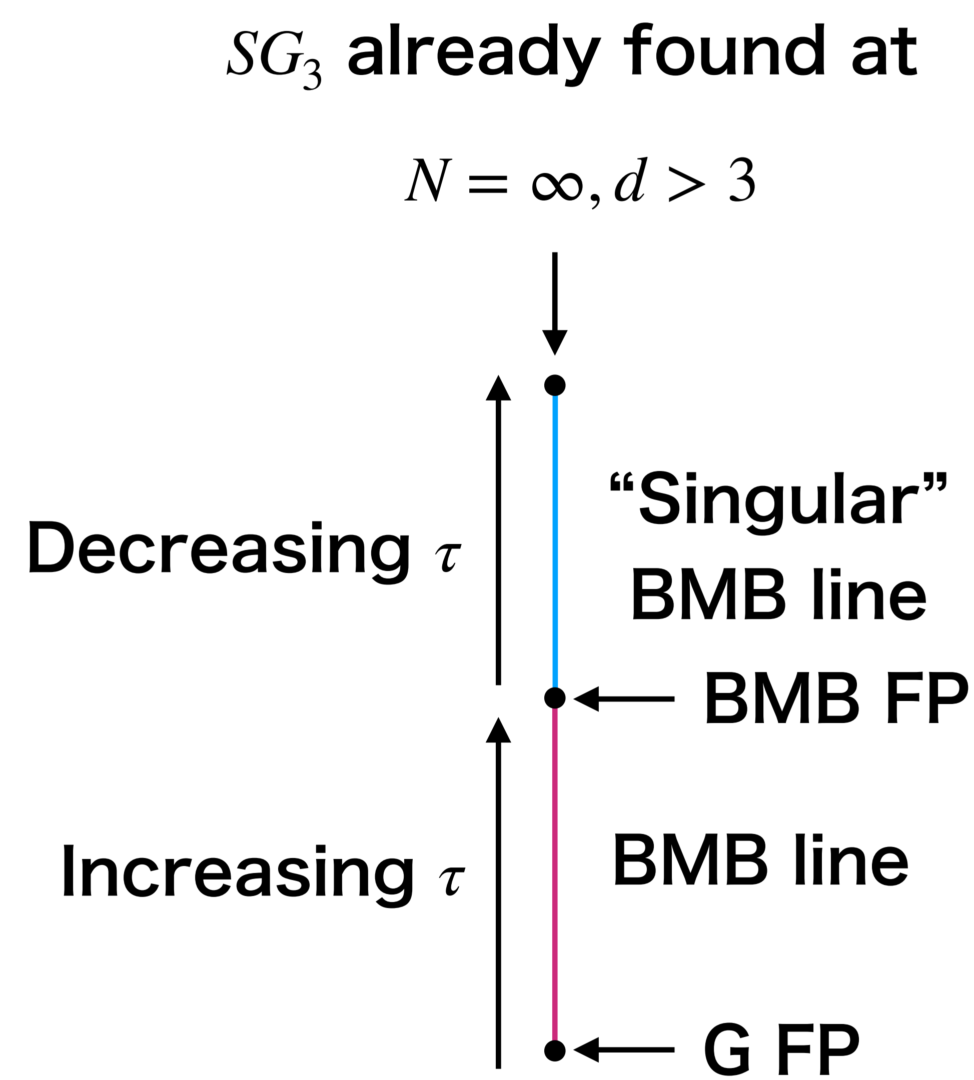
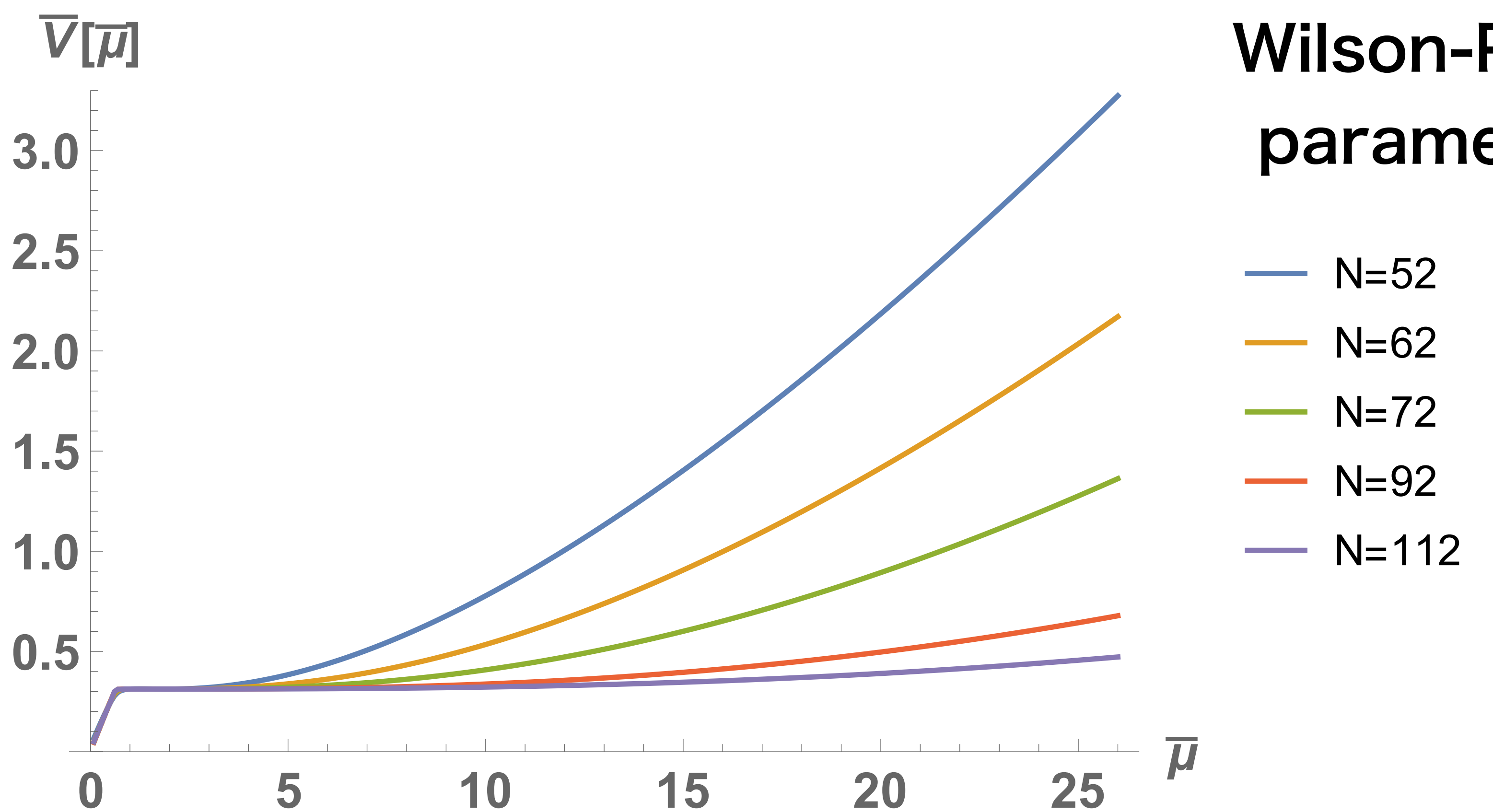


FIG. 2. $N = \infty$ and $d = 3$: Singular potential of $S\mathcal{A}(\tau = 0.33)$ from the potential of $\mathcal{A}(\tau = 0.33)$ given by the red and dashed red curves, Eq. (7). The green and dashed green curves show $\bar{V}(\bar{\varrho}) = \bar{\varrho}$. The potential of $S\mathcal{A}(\tau = 0.33)$ is made of the plain green and red curves that meet at $\bar{\varrho}_0(\tau = 0.33)$. Inset: zoom of the region around the cusp and its rounding at finite N within the boundary layer.



SG_3 (with 3 relevant directions) in $3 < d < 4$

SG_3 **in d=3.2**

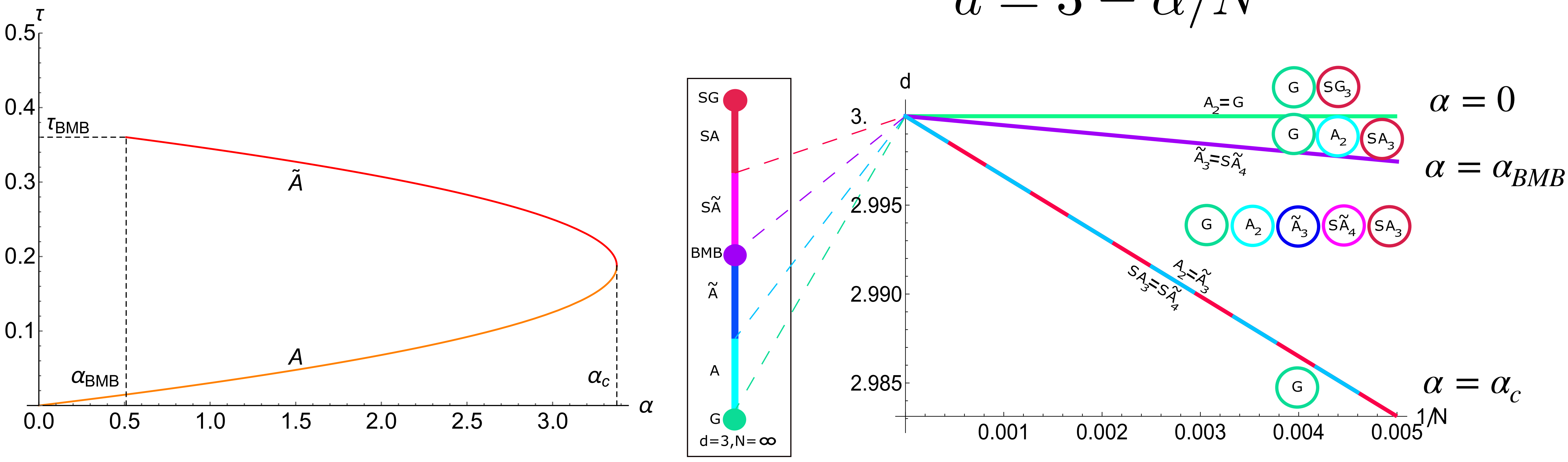


Wilson-Polchinski parametrization

PRL 121, 231601 (2018)

Fixed point structure in the vicinity of $d=3, N=\infty$

$$d = 3 - \alpha/N$$



- The number of relevant directions around a FP at finite but large N is indicated with the subscript.

Summary

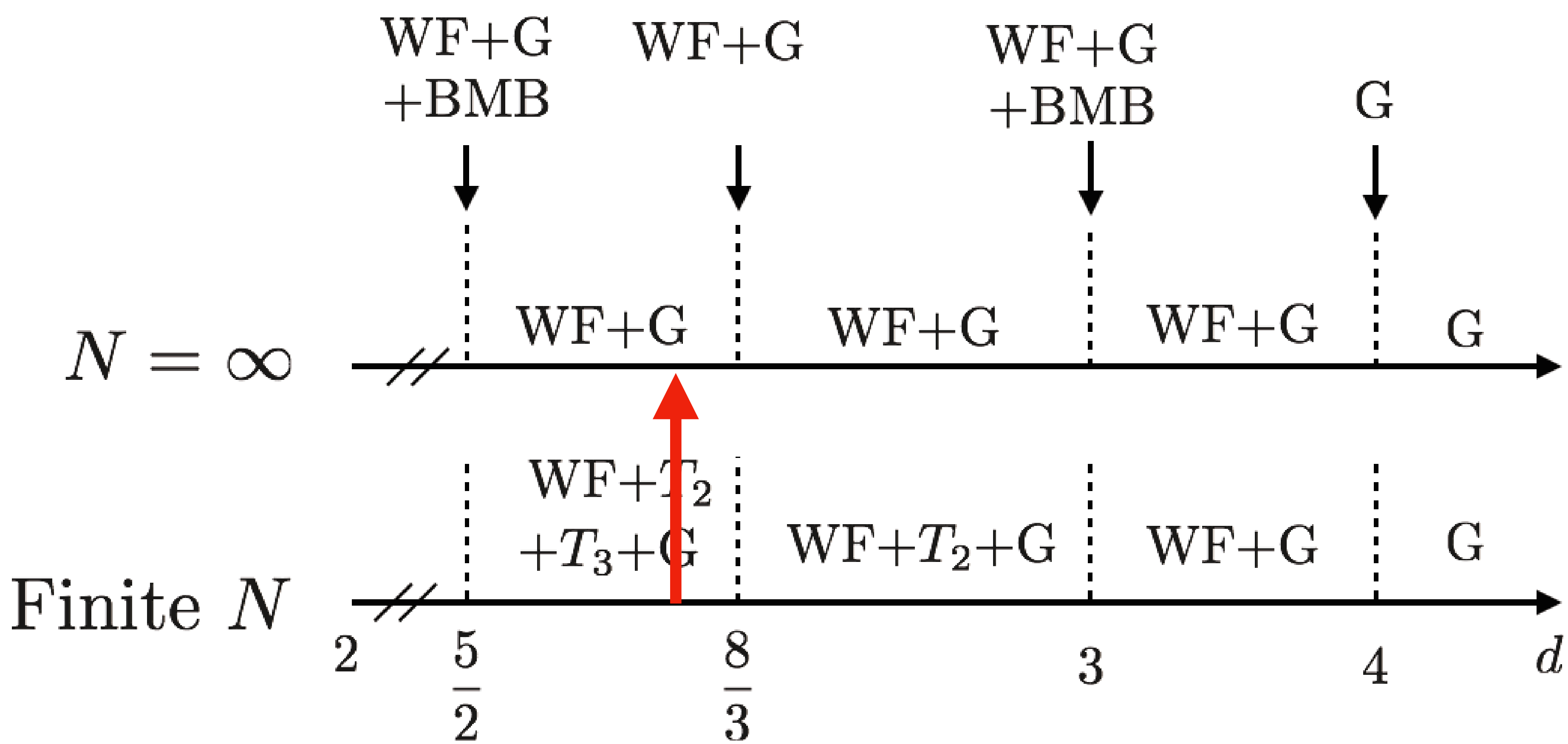
- The large N limit that allows us to find the BMB line must be taken on particular trajectories in the (d, N) plane: $d = 3 - \alpha/N$ and not at fixed dimension d= 3.

- Our study also reveals that the known BMB line is only half of the true line of fixed points, the second half being made of singular fixed points.

- The potentials of these singular fixed points show a cusp for a finite value of the field and their finite N counterparts a boundary layer.

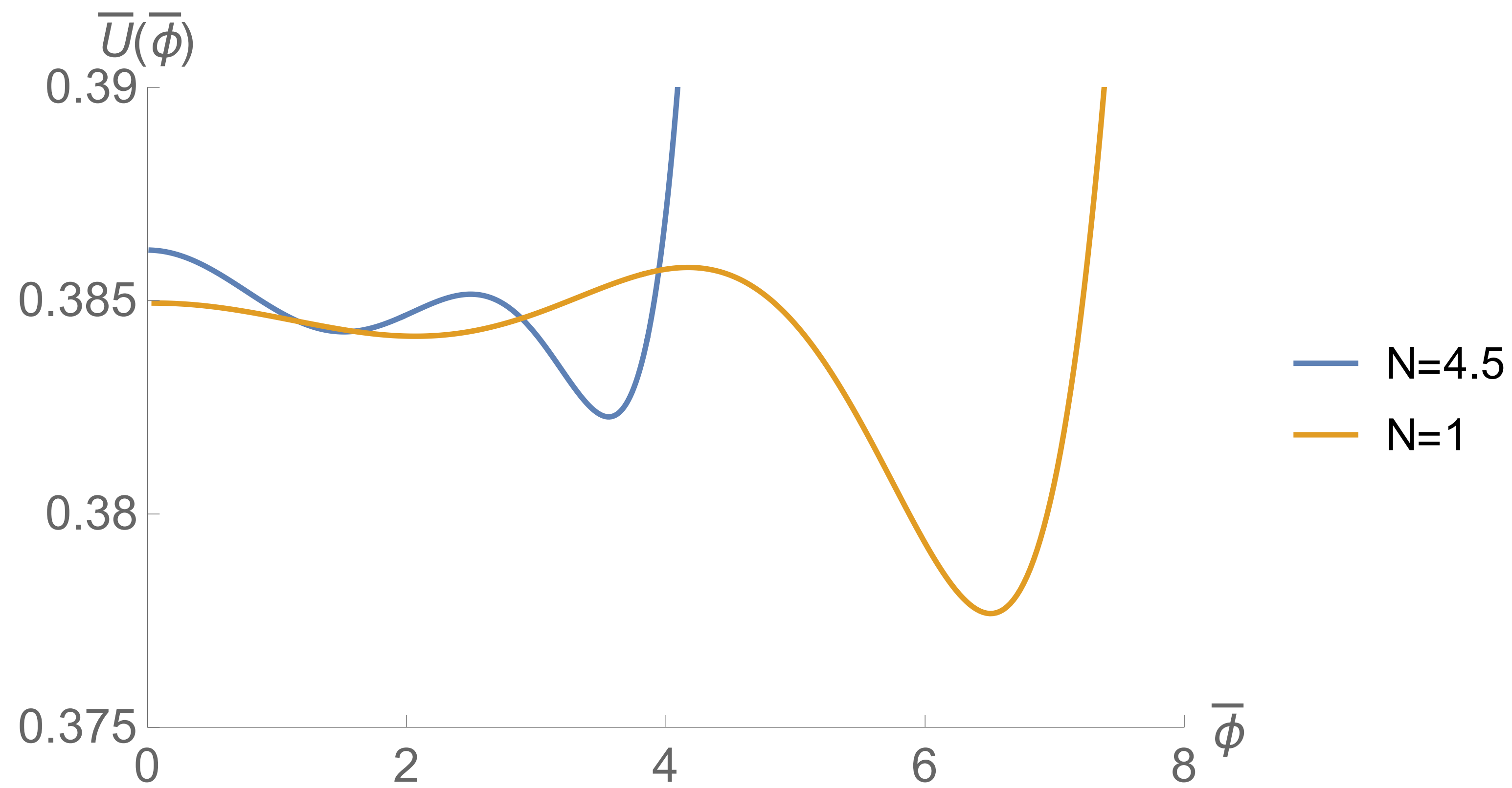
Results on the BMB line at $d = 8/3$ and $N = \infty$

Simple paradox on the tetracritical FP T_3



- What occurs if we follow T3 from $(d = \frac{8}{3}^{-}, N = 1)$ to $(d = \frac{8}{3}^{-}, N = \infty)$ continuously as a function of (d,N)?

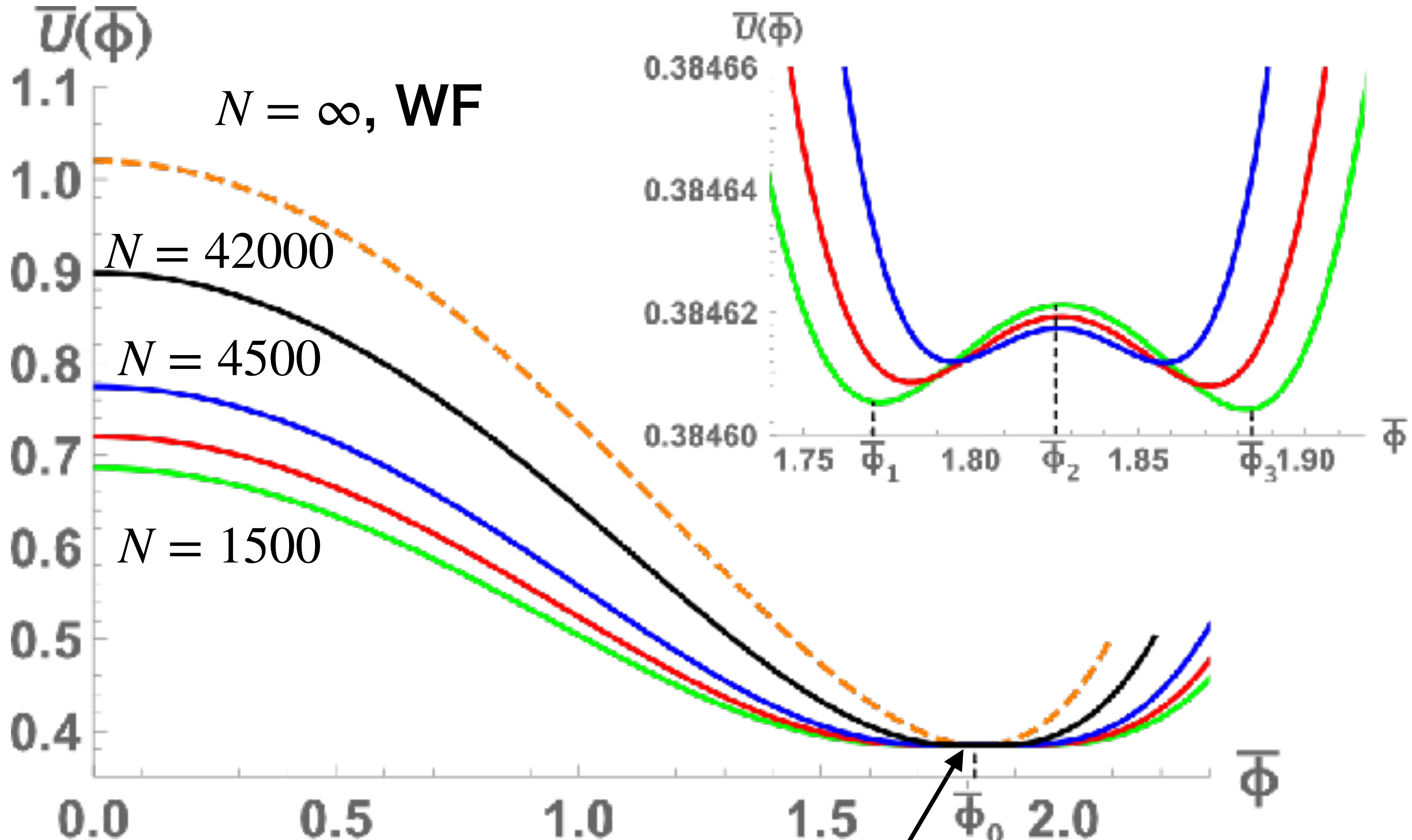
T3 in d=2.6 for small N



T3 has three extrema in $\bar{\phi} > 0$.

The three extrema approach for larger N

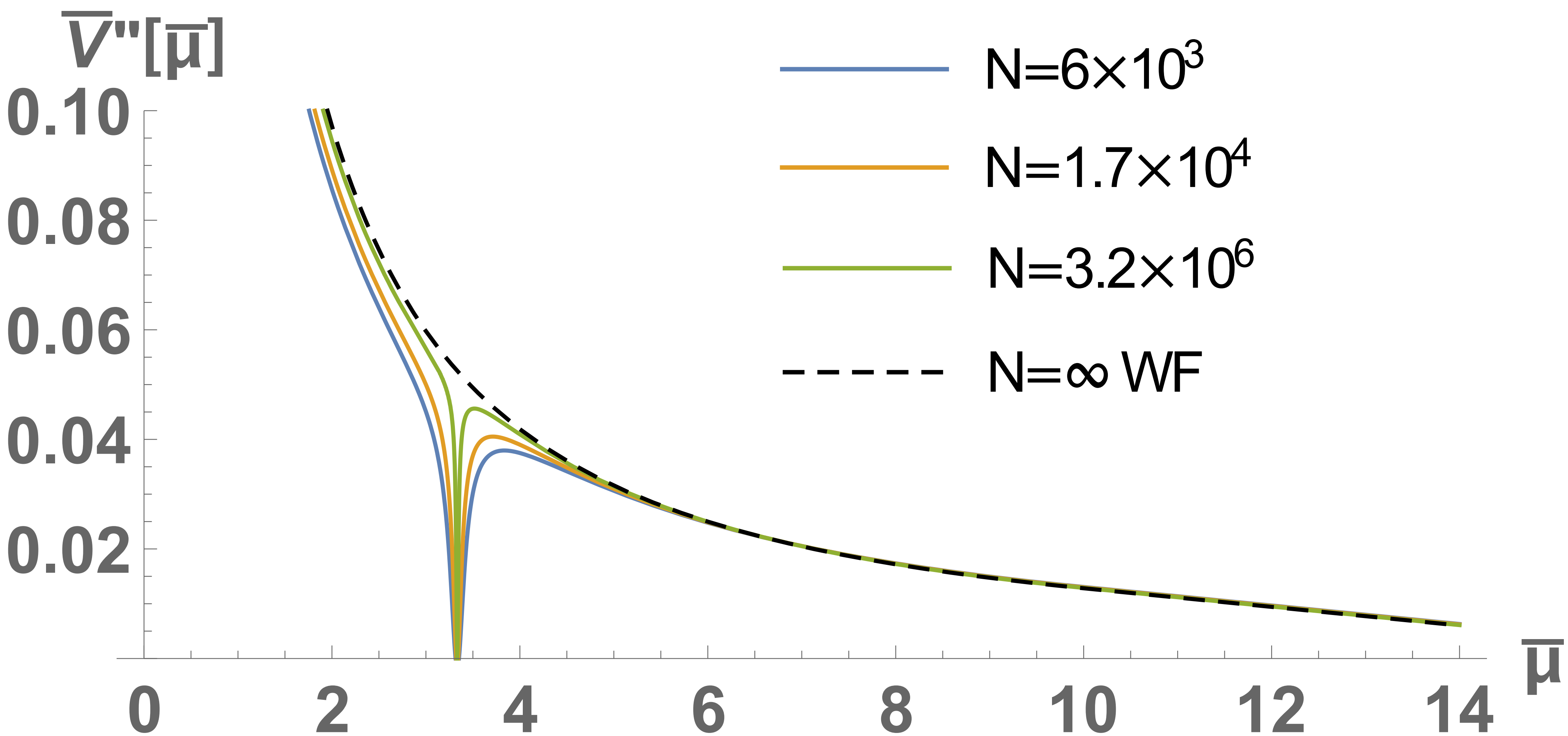
T_3 in d=2.6



The potential becomes very flat,
since three extrema become very close.

Numerically T_3 continues to exist up to N=∞.
Why is the Large-N limit not captured by conventional Large-N analysis??

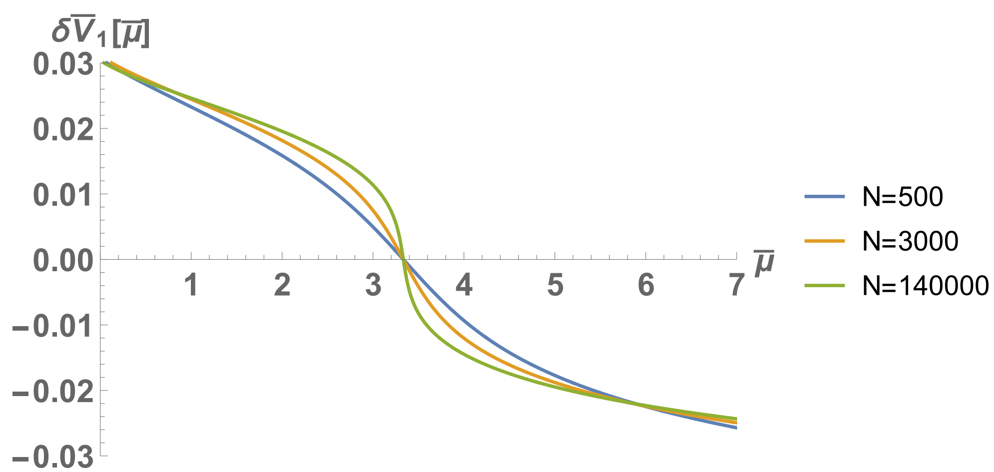
Global plot of the second derivative of the potential

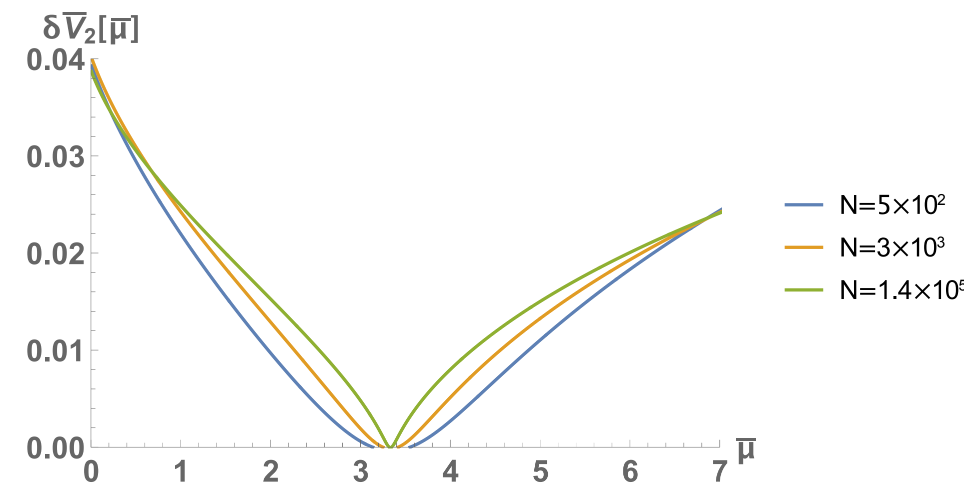


It converges to that of WF FP except at $\bar{\mu} = \bar{\mu}_0$

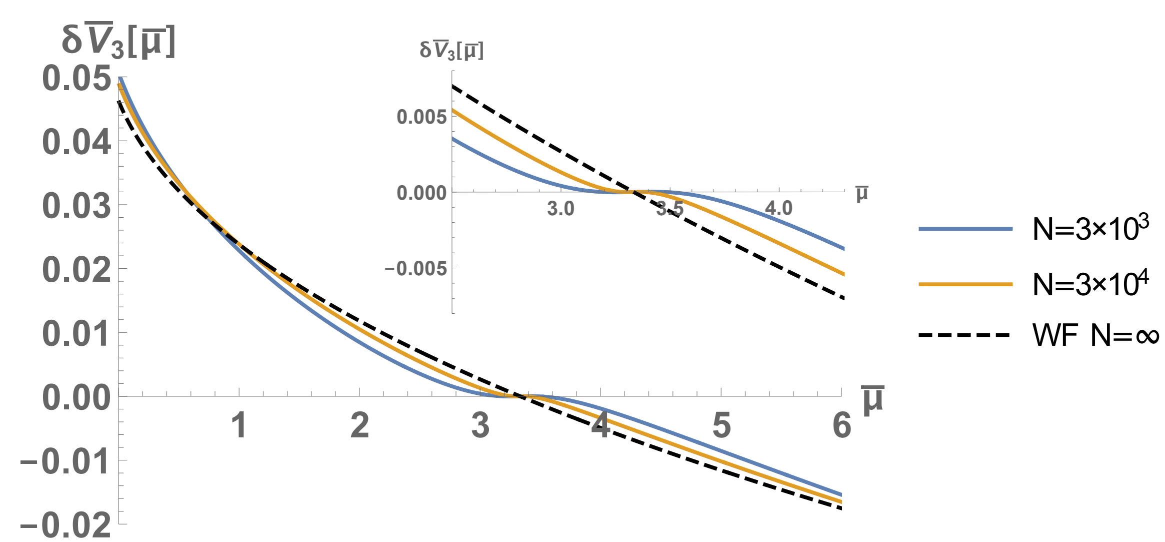
The difference between T_3 and WF can be seen not in their potential but in their derivatives

Eigenperturbations around T_3





The two eigenperturbations become singular.



The corresponding eigenvalue tends to $\nu^{-1} = d - 2$

Scaling behavior inside the boundary layer

- For very large N, the distances between the three extrema are proportional to $N^{-1/2}$.

- $\bar{U}''(\bar{\phi})$ at the three extrema approach constant values. $\cdots$The third and higher order derivatives become singular.

- We can expect a scaling $\bar{U}''\left(\bar{\phi}\right) \simeq f\left(N^{1/2}\left(\bar{\phi}-\bar{\phi}_0\right)\right)$.

- We can identify the position of the boundary layer as $\bar{\phi}_0 \simeq \sqrt{2/(d-2)}$, from numerical solutions and boundary layer analysis

Boundary layer analysis

- To simplify the notation we employ Wilson-Polchinski version of LPA FP eq.

$$0 = 1 - d\,\bar{V} + (d-2)\bar{\mu}\bar{V}' + 4\bar{\mu}\bar{V}'^2 - 2\bar{V}' - \frac{4}{N}\bar{\mu}\,\bar{V}''$$

- Around $\tilde{\mu} = N^{1/2}(\bar{\mu} - \bar{\mu}_0)$

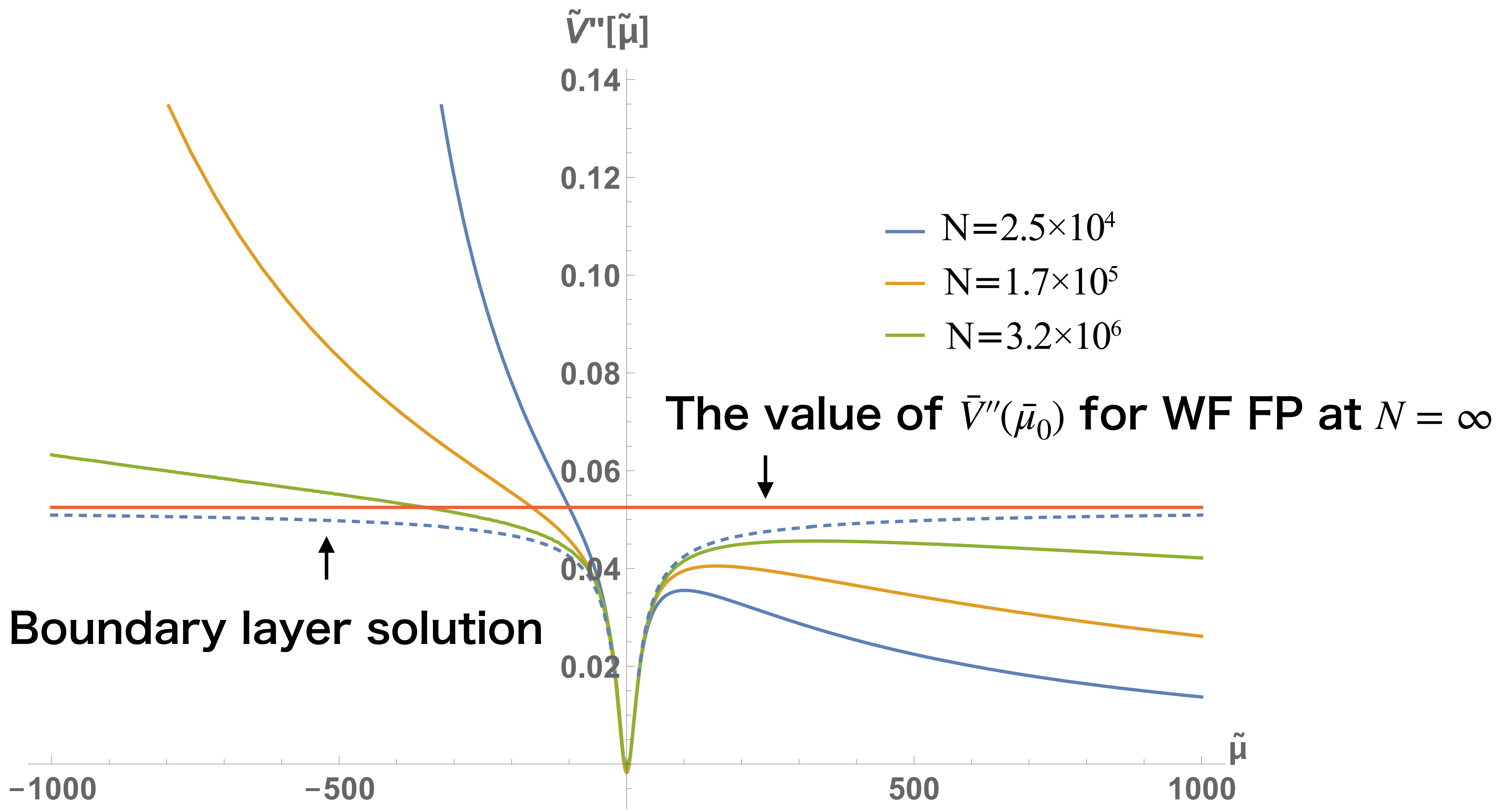
  we introduce a scaled variable $\tilde{\mu} = N^{1/2}(\bar{\mu} - \bar{\mu}_0)$,

  and the potential is scaled as $\tilde{V}_N(\tilde{\mu}) = N\left(\bar{V}(N^{-1/2}\tilde{\mu} + \bar{\mu}_0) - 1/d\right)$.

- The $O(\epsilon)$ contribution of FP eq. vanishes if we set $\bar{\mu}_0 = 2/(d-2)$, and the $O(\epsilon^2)$ contribution is

$$-\frac{8\tilde{V}''_\infty(\tilde{\mu})}{d-2} + \frac{8\tilde{V}'_\infty(\tilde{\mu})^2}{d-2} + (d-2)\tilde{\mu}\tilde{V}'_\infty(\tilde{\mu}) - d\tilde{V}_\infty(\tilde{\mu}) = 0$$

Scaled boundary layer for finite but very large N

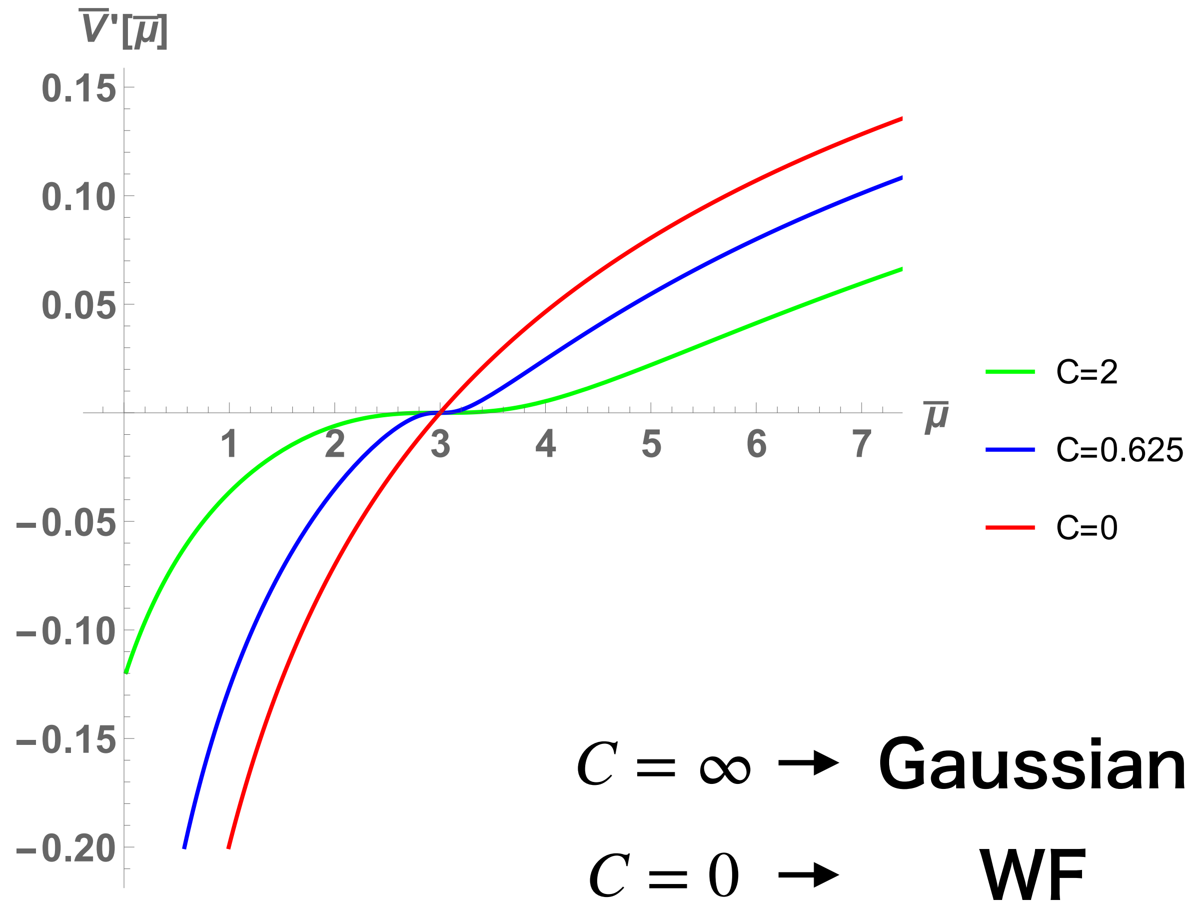


BMB line

in $d = 8/3$ and at $N = \infty$

$$\bar{\mu}_{\pm} = \frac{C}{\bar{V}'\left(1 - 2\bar{V}'\right)}\left(\frac{\pm 2\bar{V}'}{1 - 2\bar{V}'}\right)^{4/3} + 2f(4\bar{V}')$$

$$f(x) = \frac{3}{2 - x} + \frac{4x}{(2 - x)^{7/3}}\int_0^1 dz \left(\frac{2 - xz}{z}\right)^{1/3}$$



$C = \infty$ → **Gaussian**

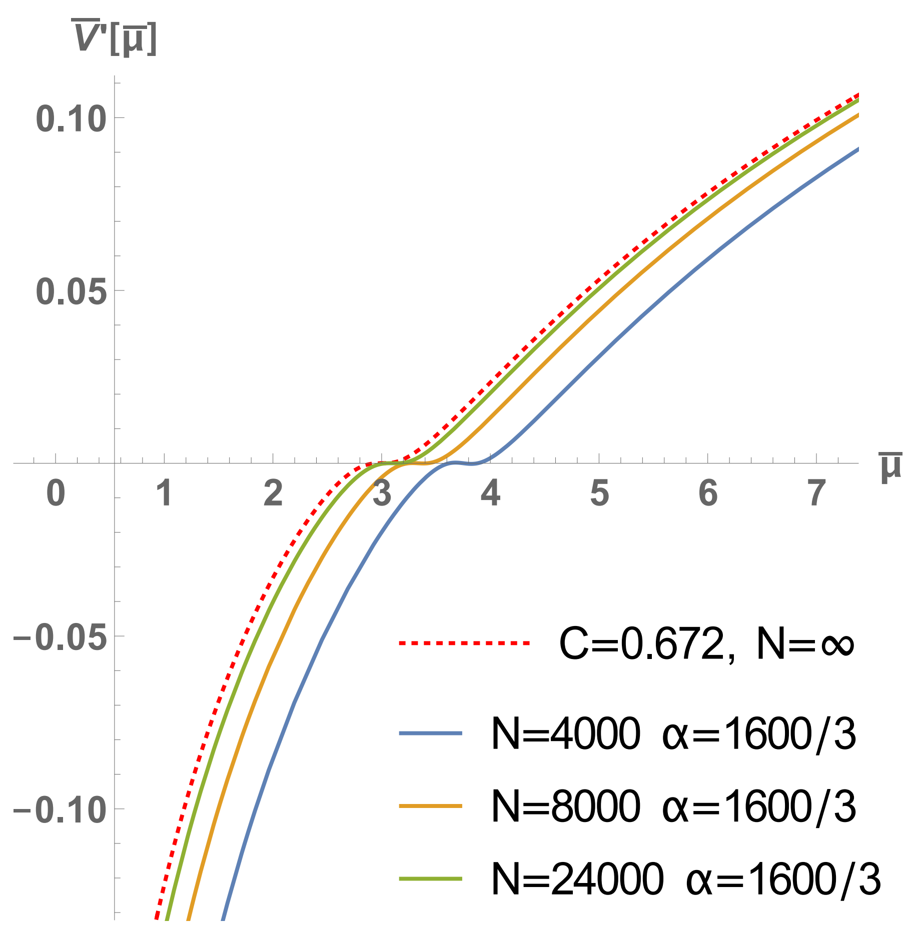
$C = 0$ → **WF**

Here taking derivatives (at $\bar{\mu} = 3$) and the limit $C \to 0$ do not commute, which explains the difference of the stability between T_3 and WF

Finite-N analysis around

$d = 8/3, N = \infty$

- When we follow T_3 on the hyperbola $\epsilon N = \alpha$=const, T_3 converges to a FP on the BMB line $\alpha = 162/C^3$.



Summary

- We followed tetracritical FP T_3 in O(N) models increasing N with LPA.
- It seems that T_3 continues to exist up to N=∞. The third and higher order derivatives become singular at $\bar{\rho} = \bar{\rho}_0$.
- The potential converges to that of WF FP except at $\bar{\rho} = \bar{\rho}_0$.
- Can we conjecture that a similar scenario holds for T_n with odd n?

$O(N) \times O(2)$ model

- Order parameter in the ground state… a $N \times 2$ matrix $\Phi = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$

  that satisfies $\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = \delta_{ij}$

- Ginzburg-Landau-Wilson Hamiltonian is given as

$$H = \int d^d\mathbf{x} \left(\frac{1}{2} \left[(\partial \boldsymbol{\phi}_1)^2 + (\partial \boldsymbol{\phi}_2)^2 \right] + U(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) \right)$$

with $U(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ which takes a minimum when $\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = const \times \delta_{ij}$

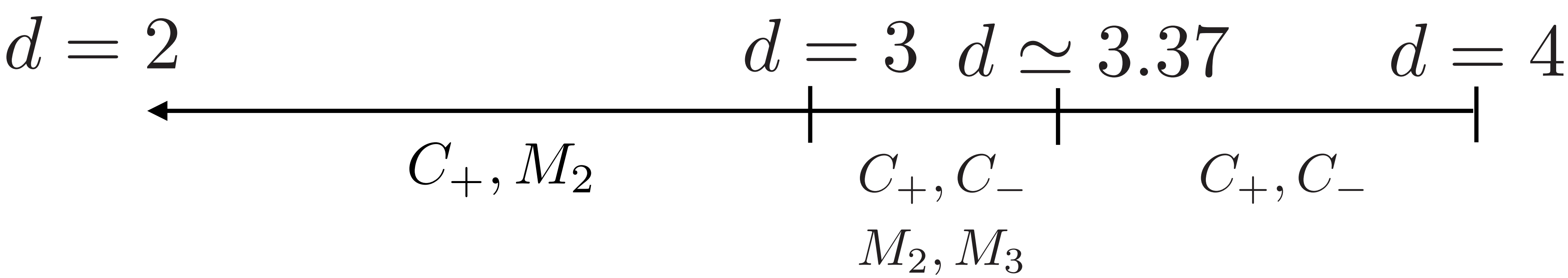
$$\tilde{U}_k\left(\tilde{\psi}_1, \tilde{\psi}_2\right) = \sum_{n,m=0} \frac{1}{n!m!} \tilde{a}_{n,m} (\tilde{\rho} - \tilde{\kappa})^n \tilde{\tau}^m$$

$O(N) \times O(2)$ invariants…

$$\begin{aligned} \rho &= \mathrm{Tr}\left({}^t\Phi\Phi\right) \\ \tau &= \tfrac{1}{2}\mathrm{Tr}\left({}^t\Phi\Phi - \rho/2\right)^2 \end{aligned}$$

$O(N) \times O(2)$ model

$N = \infty$



$$\begin{aligned} \tilde{\kappa} &= N^{-1} K^{-}, \\ \tilde{a}_{0,l} &= N^{-2l+1} A^{-}_{0,l}, \\ \tilde{a}_{k,l} &= N^{-k-2l} A^{-}_{k,l} \quad (\text{for } k \neq 0)\,, \end{aligned}$$

C- FP

C+ FP

(usual scaling)

$$\begin{aligned} \tilde{\kappa} &= N^{-1} K, \\ \tilde{a}_{k,l} &= N^{-k-2l+1} A_{k,l} \end{aligned}$$

$$\begin{aligned} \tilde{\kappa} &= N^{-1} K^{--(---)}, \\ \tilde{a}_{0,1} &= N^{-1} A^{--(---)}_{0,l}, \\ \tilde{a}_{k,l} &= N^{-\frac{k}{2}-\frac{3l}{2}} A^{--(---)}_{k,l} \quad (\text{for } (k,l) \neq (0,1)) \end{aligned}$$

M2,M3 FP

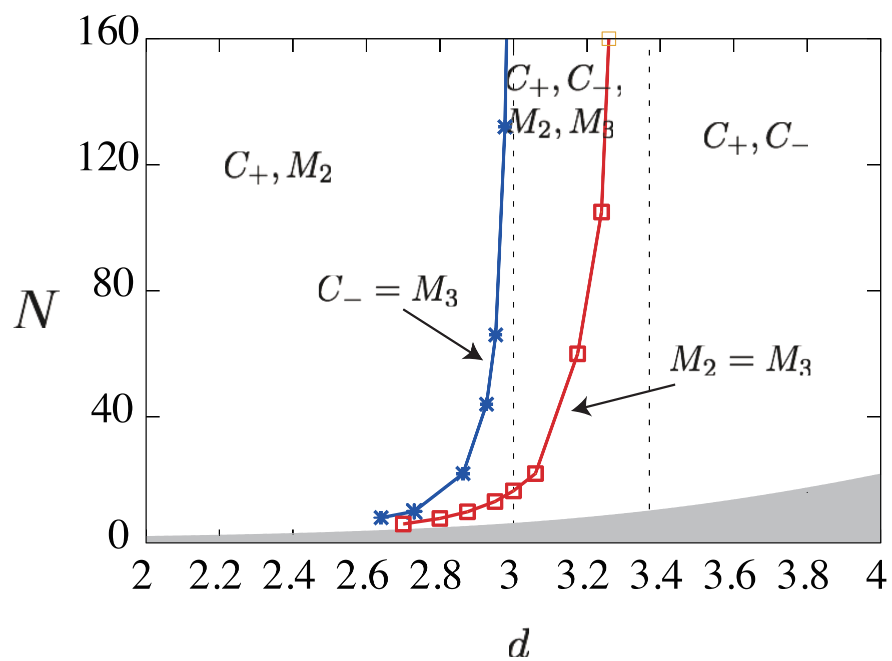


FIG. 4. O(N)⊗O(2) model. In the gray region, starting in $d = 4$ at $N = 21.8$, no FP at all is found. Above this region and for d close to 4, both the critical C_+ and the tricritical C_- FPs are found. The line on the right joining the squares indicates the region where two nonperturbative FPs, M_2 and M_3, appear. On the line joining the crosses, C_- and M_3 collapse. In each region, we indicate the FPs that are present.